The Rosseland Approximation

In the interiors of stars, at large optical depths from the photosphere, the mean free path of photons is generally very small compared to any scale lengths of gradients in the physical conditions like temperature or density. In such a situation, matter and photons are very close to the conditions of local thermodynamic equilibrium (LTE), and the source function S_ν as well as the local intensity I_ν are both very similar to the blackbody function $B_\nu(T)$. Furthermore, the mean free path of photons is much smaller than the radius of curvature of surfaces of equal temperature and density, so that a plane-parallel approximation,

$$\frac{\mu}{\alpha_\nu}\,\frac{dI_\nu}{dz} = S_\nu - I_\nu \tag{1}$$

can be used, where $\alpha_\nu = \kappa_\nu\,\rho$ is the total (true absorption + scattering) absorption coefficient. Under these conditions, one can define an appropriate average of the absorption coefficient (the Rosseland mean), and derive a simple expression for the local photon flux as a function the temperature gradient and the Rosseland mean opacity.

First, we re-write Eq. (1) as

$$I_\nu = S_\nu - \frac{\mu}{\alpha_\nu}\,\frac{dI_\nu}{dz}. \tag{2}$$

Because the system is very close to LTE, a reasonable (0th order) first guess for the I_ν is

$$I_\nu^{(0)} \approx S_\nu \approx B_\nu(T) \tag{3}$$

and the second term on the r.h.s. of Eq. (2) is small, so that we can replace I_ν by $B_\nu(T)$ to get a first-order approximation. Thus,

$$I_\nu^{(1)} \approx B_\nu(T) - \frac{\mu}{\alpha_\nu}\,\frac{\partial B_\nu(T)}{\partial z} \tag{4}$$

Now, we can insert Eq. (4) into the general expression for the radiative flux to obtain

$$F_\nu = \int\limits_{4\pi} I_\nu\,\mu\,d\Omega = -\frac{2\pi}{\alpha_\nu}\,\frac{\partial B_\nu(T)}{\partial z}\int\limits_{-1}^{1}\mu^2\,d\mu = -\frac{4\pi}{3\,\alpha_\nu}\,\frac{\partial B_\nu(T)}{\partial z} \tag{5}$$

Note that the flux from the first term on the r.h.s. of Eq. (4) is 0 because $B_\nu(T)$ is isotropic. Furthermore, $B_\nu(T)$ depends on depth z only through variations of the temperature, so that we may write

$$F_\nu = -\frac{4\pi}{3\,\alpha_\nu}\,\frac{\partial B_\nu(T)}{\partial T}\,\frac{\partial T}{\partial z}. \tag{6}$$

Finally, in order to obtain an expression for the total flux, we integrate Eq. (6) over all frequencies:

$$F = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int\limits_0^\infty \frac{\partial B_\nu(T)}{\partial T}\,\frac{1}{\alpha_\nu}\,d\nu. \tag{7}$$

This result motivates a definition of an average absorption coefficient (or, equivalently, an average opacity), α_R, called the **Rosseland Mean**: We require that the integral on the r.h.s. of Eq. (7) is

$$\int\limits_0^\infty \frac{\partial B_\nu(T)}{\partial T}\,\frac{1}{\alpha_\nu}\,d\nu \equiv \frac{1}{\alpha_R}\int\limits_0^\infty \frac{\partial B_\nu(T)}{\partial T}\,d\nu. \tag{8}$$

Now, we know that

$$\int\limits_0^\infty B_\nu(T)d\nu = \frac{\sigma}{\pi}T^4 \tag{9}$$

so that

$$\int\limits_0^\infty \frac{\partial B_\nu(T)}{\partial T}\,d\nu = \frac{\partial}{\partial T}\int\limits_0^\infty B_\nu(T)\,d\nu = \frac{4\sigma T^3}{\pi}. \tag{10}$$

Thus, we obtain for the **Rosseland Mean:**

$$\frac{1}{\alpha_R} = \frac{\pi\int_0^\infty \frac{\partial B_\nu(T)}{\partial T}\frac{1}{\alpha_\nu}\,d\nu}{4\,\sigma\,T^3} \tag{11}$$

and for the **Rosseland approximation for the total flux:**

$$F = -\frac{16}{3}\,\sigma\,T^3\,\alpha_R^{-1}\,\frac{\partial T}{\partial z}. \tag{12}$$