Exercise on Synchrotron

Taken from Rybicki & Lightman, Problem 6.1

Synchrotron energy loss

An ultrarelativistic electron emits synchrotron radiation.

a. Show that its energy decreases with time according to:

$$\gamma = \gamma_0 (1 + A\gamma_0 t)^{-1} \tag{1}$$

where A is a constant defined as:

$$A = \frac{2e^4 B_\perp^2}{3m^3 c^5} \tag{2}$$

Here γ_0 is the initial Lorentz factor at time $t = t_0$ and $B_\perp = Bsin\alpha$ with α the pitch angle between electron velocity vector and $\vec{B}$.

b. Show that the time t_h for the electron to lose half its energy is:

$$t_h = \frac{1}{A\gamma_0} \tag{3}$$

Solution

a. On the ground of energy conservation we have $P = -dE/dt$. We know that:

$$P = \frac{2e^4 \gamma^2 \beta^2 B_\perp^2}{3m^2 c^3} \tag{4}$$

since $\beta_\perp B = B_\perp \beta$. The relativistic energy is given by γmc^2. Taking the derivative we have:

$$-\frac{d(\gamma mc^2)}{dt} = A\beta^2\gamma^2 mc^2 \tag{5}$$

This can be rewritten as:

$$\dot{\gamma} = -A\beta^2\gamma^2 \tag{6}$$

Since the electron is ultrarelativisitc, then $\beta \approx 1$. Therefore:

$$\frac{d\gamma}{\gamma^2} = -Adt \tag{7}$$

Integrating gives:

$$\gamma = \frac{1}{At - C} \tag{8}$$

where C is the constant of integration, that can be set as initial condition $C = -\gamma_0^{-1}$. Therefore:

$$\gamma = \frac{\gamma_0}{1 + A\gamma_0 t} \tag{9}$$

b. At t_h the Lorentz factor must have halved so that $\gamma = \gamma_0/2$. Therefore $\gamma/2 = \gamma_0/(1 + A\gamma_0 t_h)$. Solving for t_h gives:

$$t_h = \left(A\gamma_0\right)^{-1} \tag{10}$$