Clarification on the Radiation Pattern and Measured Electric Field of an Accelerating Charge

Notes on Lecture 6

Question

Today someone asked the following: if an observer is at an arbitrary distance from an electron accelerated in circular motion, why the measured electric field is sinusoidal? Indeed the radiation pattern has a $\sin^2\Theta$ dependency (doughnut shape), so that it is not very clear why the electric field should be sinusoidal.

Explanation

The answer is that at close distances the electric field is not sinusoidal indeed. $\vec{E}(t)$ becomes sinusoidal only sufficiently far from the charge. How far? Far enough so that the acceleration field E_{rad} becomes dominant over the velocity field.

Let's demonstrate first why the measured electric field is sinusoidal at large distances. The doughnut radiation pattern gives the power emitted per unit solid angle (see Figure 1), $\frac{dP}{d\Omega}$ (or $\frac{dW}{dtd\Omega}$ in the slides, with W as symbol for energy). We found that:

$$\frac{dP}{d\Omega} = \frac{\ddot{d}}{4\pi\, c^3}\sin^2\Omega. \tag{1}$$

and $\ddot{d}$ the second time derivative of the dipole moment $d = e\vec{r}$. Let's go backward on how we derived this formula.

We started from the pointing vector S, which has units of flux (erg/s/cm^2, or power/area) and we needed to transform it into a power per unit solid angle. To do that we needed to multiply S by an area, that we can choose from the definition of solid angle $dA = d\Omega R^2$. Indeed the Poynting flux is:

$$S = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Omega \tag{2}$$

But this is nothing but:

$$S = \frac{c}{4\pi}E^2_{\mathrm{rad}} \tag{3}$$

So the Poynting flux is just the square of the *radiation* electric field (or acceleration field). However, we know that this is the dominant part of the field only far away from the charge. Indeed the acceleration field magnitude scales as $1/R$, whereas the velocity field scales as $1/R^2$ (remember the red bars on the animation with the yellow hue being the Poynting vector, see Figure 2)

As you see from the figure, the velocity field is dominant near the charge. Therefore the radiation pattern near the charge, where $E_{vel} \gg E_{acc}$ will not be a doughnut shape. Indeed the Poynting flux will not be the one described in the Eq. 3 because that was derived assuming $E_{rad} \gg E_{vel}$. Therefore the electric field will not be sinusoidal and there will be

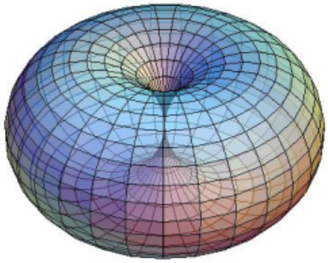

Figure 1. The dipole radiation field pattern of an accelerating charge. The direction of the acceleration is along the "hole" of the doughnut.

multiple harmonics of the cyclotron line (i.e., the power spectrum will contain multiple frequencies).

In other words, the radiation pattern will not be a doughnut shape (indeed the red lines closest to the charge in Figure 2 are not orthogonal to the electric field vector, because of

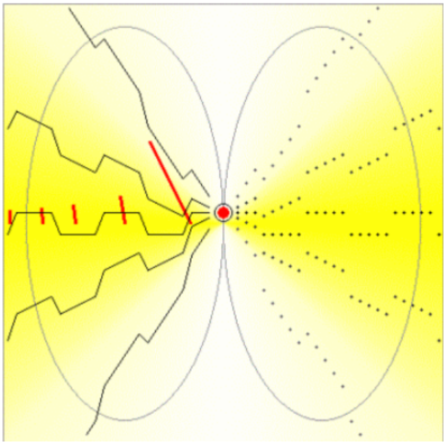

Figure 2. The figure is a snapshot of an animation shown in class. A charged particle is switched up and down in a very strong electric field, such that the shape is traced out in time and aligns to an approximate square wave. The ovals' reference lines are drawn to the left and right of the charge and correspond to a cross-section through the doughnut toroid, as illustrated in the previous diagram. Based on the criteria of the Larmor formula, when a charge is subject to acceleration, i.e. during the transition positions, it radiates power also subject to the angle θ with respect to the axis of charge motion. As such, the energy density is reflected by the depth of the yellow shading, symmetrical about the axis of motion. The oscillating red lines on the left reflect the total electric field $E = E_{rad} + E_{vel}$ as a function of distance. So what you see is the effects of E_{vel} reducing by $1/R^2$, while E_{rad} only reduces by $1/R$ and so quickly becomes the dominant field as the radius from the charge increases.

the dominant velocity field component).