Astrophysical Radiative Processes

* D R A F T *

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In astrophysics, photons are the main carriers of information. The amount of energy carried by photons over time defines the field of photometry, an area of astrophysics that deals with the flux and intensity of objects and their variation over time. The distribution of the photon frequencies defines the energy spectra that, in turn, convey information on the radiative processes that generated them. It is also possible to extract information on the physical system under observation by measuring the time of arrival of photons, a field known as timing. Finally, by measuring the spin of photons (polarization) one can understand how they were generated, the scattering processes and determine the presence of magnetic fields in the environment where photons propagate.

It is thus of great importance to understand first how photons propagate in space, how they interact with matter and fields and understand which physical processes generate them.

To begin with, we need to specify some definitions that will be useful throughout the book. These definitions are necessary because we will deal with quantities which are rigorous and have a specific meaning in order to avoid confusion.

The first definition that we need to introduce is that of a light ray. A photon has a wavelength:

$$\lambda = \frac{c}{\nu} \tag{1.1}$$

where c is the speed of light and ν the photon frequency. The energy of the photon is therefore:

$$E = h\nu = \hbar\omega \tag{1.2}$$

where h is the Planck constant, $\hbar = h/2\pi$ and $\omega = 2\pi\nu$. From here we can define a relation between wavelength and momentum of a photon:

$$\lambda = \frac{h}{p} \tag{1.3}$$

where p is the momentum. When the wavelength of light becomes shorter than a few tens of nm it is convenient to switch units and use the electron-Volt (eV), which is the reference unit for X-ray and γ-rays (at least in astrophysics). The eV is not part of the Standard International system of units (SI) since it is an energy (and the SI energy unit is the Joule). The reason why this unit is used will be clear later. Once we have defined the energy, frequency and wavelength of photons we immediately see a first problem: if the wavelength of light is λ, then, according to the Heisenberg principle, the direction of the photon, encoded in its momentum p, has an uncertainty which is inversely proportional to the square of the photon wavelength. This implies that the concept of a specific direction of a photon breaks down in certain circumstances. When this is not the case, a useful approximation is made by defining the concept of rays, which are physical entities associated with photons. A bundle of rays can be thought of as a set of quasi-plane waves that can have any direction at each point and that propagate in space by following the laws of geometric optics. The bundle of rays should be considered as a superposition of incoherent waves, in the sense that no interference effect are present and their intensity is the sum of the intensity of each wave without amplitude terms.

1.1 Definitions

Specific Brightness. The fundamental concept associated with radiative transfer theory is that of intensity, or brightness. The word *specific* that sometimes appears associated with these terms means that the intensity is defined per unit frequency (i.e., monochromatic light). To arrive at a rigorous definition of specific brightness we follow a bundle of rays passing through a given area. We first pick a single ray with frequency ν, then we consider an infinitesimal area dA normal to the direction of the given ray $\vec{n}$ and we consider all rays passing through dA whose direction is within an infinitesimal solid angle $d\Omega$ of the given ray. The *specific intensity* is then defined in terms of the infinitesimal energy dE carried by the rays:

$$I_\nu = \frac{dE}{dA\, d\Omega\, dt\, d\nu} \ . \tag{1.4}$$

The specific intensity is therefore the infinitesimal energy carried by a bundle of rays with frequency between ν and $\nu + d\nu$ propagating within a solid angle $d\Omega$ and crossing an infinitesimal area dA in an infinitesimal time dt.

Specific Flux. From the definition of specific intensity above one might notice its resemblance with that of specific flux, with one crucial difference. Since the flux is an energy per unit time per unit area, we can define the specific flux as:

$$F_\nu = \frac{dE}{dA\,dt\,d\nu} \ . \tag{1.5}$$

We see that the difference between Eq. (1.5) and Eq. (1.4) is just given by the presence of the solid angle $d\Omega$. What is the reason of this difference and what is the physical meaning of these two quantities? The presence of the unit solid angle tells us that the specific brightness requires that our bundle of rays are coming from a direction normal to dA within a solid angle $d\Omega$ from the direction $\vec{n}$. Therefore the source emitting the bundle of rays needs to be *resolved*, which means that different rays crossing the surface area dA and forming our bundle, come from slightly different directions from slightly different points belonging to the source. In other words, a point source cannot have a specific intensity, since all rays come from a single direction. In that case we would talk about specific flux rather than intensity.

Radiation Pressure. If we have rays moving in all directions then we have a radiation field. In this case we can define momentum flux, useful to later discuss radiation pressure. The pressure of radiation at a certain infinitesimal area (see Fig 1.1 is calculated from the net rate of transfer of momentum normal to that area dA.

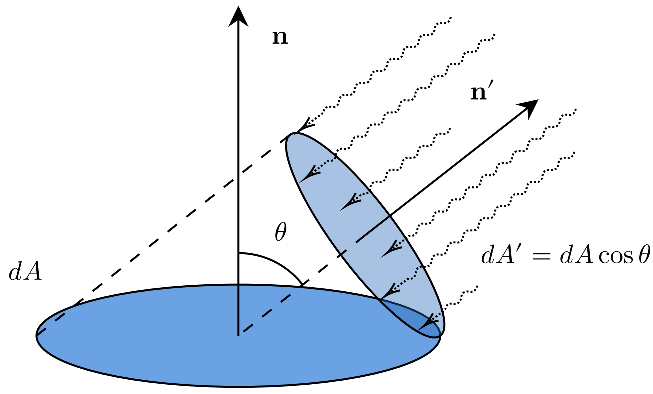

Figure 1.1: Bundle of rays.

The first step is to calculate the momentum flux *normal* to dA carried by radiation coming from a direction $\vec{n}'$ which is *different* from the normal $\vec{n}$ to the area dA. In the figure the rays are the dot-dashed red arrows coming form a direction $\vec{n}''$. Why do we choose a different direction $\vec{n}' \neq \vec{n}$? Simply because we are discussing the general case to derive the expression for the radiation pressure and the case of radiation coming from a direction $\vec{n}$ is comprised in this general case.

When we say that radiation is coming from $\vec{n}'$ what it really means is that the rays are coming from within a solid angle $d\Omega$, so that as usual we are considering a little bunch of rays. To make the argument as simple as possible we are referring to the direction $\vec{n}'$ but again we should think about it as meaning "rays coming from an infinitesimal solid angle $d\Omega$". Let's call the angle between $\vec{n}$ and $\vec{n}'$ as θ. These bunch of rays will not see the surface area dA, but will see a surface area $dA \cos\theta$ (i.e., the projected area).

Now let's define an expression that links the flux to surface intensity:

$$dF_\nu = I_\nu \cos\theta \, d\Omega \ , \tag{1.6}$$

$\cos\theta$ appears in the integral because of the projected area effect $dA' = dA \ \cos\theta$.

To understand the meaning of certain physical systems it is useful to think about the most extreme cases. Therefore let's say that $\theta = 0$, in which case $\cos\theta = 1$ and we are in the case $\vec{n} = \vec{n}'$. If $\theta = 90°$ then there is no flux (the rays are coming from the side of the area and they see a projected area equal to zero.

What is the momentum carried by each ray? It is E/c as we all know from basic knowledge of quantum mechanics. A momentum has units of energy $\times$ time / length. The flux *of momentum* normal to dA has units of momentum per unit time per unit area which is equivalent to pressure. The component of the flux normal to dA is $dF_\nu \, cos\theta$ (see Fig 1.1), i.e., there is another $\cos\theta$ that is used to select only the component of the momentum flux *normal* to dA.

The component of the momentum flux normal to dA is $dF_\nu \cos\theta/c$. Therefore the pressure exerted by radiation on the area dA is:

$$dP_\nu = \frac{dF_\nu \cos\theta}{c} = \frac{I_\nu}{c} \cos^2\theta \, d\Omega \tag{1.7}$$

At this point it is easy to derive an expression for the radiation pressure given by a radiation field by first integrating over all solid angles and then over all frequencies:

$$P = \frac{1}{c} \int \int_{4\pi} I_\nu \ \cos^2\theta \, d\Omega \, d\nu \tag{1.8}$$

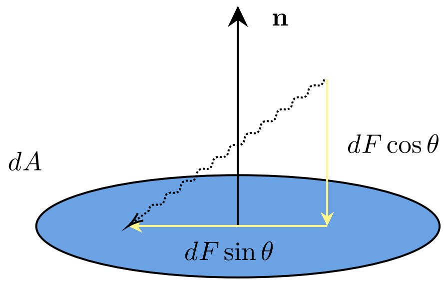

Figure 1.2:

We might now recall that pressure has the same units an an energy density u, which is energy over volume. When the radiation field is isotropic, it is easy to show that the radiation pressure is $p = u/3$. The $1/3$ can be understood in terms of symmetries. Consider two perfect mirrors perfectly parallel to each other with a photons bouncing back and forth having a direction $\vec{n}$ perfectly perpendicular to the mirrors' area. Each mirror has an area A and they are at a distance L from each other. The photons, having a total energy of, say, E, will fill a volume $A \cdot L$ and have therefore energy density E/AL. Each time a photon bounces over a mirror, it transfers a momentum $2p$ (the 2 is there because there is reflection rather than absorption of the photon, that adds another p to the initial photon momentum). The photon then crosses the length L and bounces off again and the process repeats. Since the crossing time is L/c, the photon bounces over the same mirror at an interval of $2L/c$. The momentum has units of $[M\,L\,T^{-1}]$ whereas pressure has units of energy over volume which is $[M\,L^{-1}\,T^{-2}]$. To obtain a pressure from the momentum we need to divide it by a time and an area. Therefore:

$$P = \frac{2p}{A\frac{2L}{c}} = \frac{E}{AL} = u \tag{1.9}$$

This is the pressure exerted by photons over an area perpendicular to their momentum. What about the pressure of the isotropic radiation field? In this case we can use the symmetry of the problem to show that $P = u/3$. Indeed if the radiation field is isotropic, then there is a $\cos\theta$ that needs to be taken into account in the momentum transferred *and* another $\cos\theta$ in the expression for the travel time $2L/c\cos\theta$. Since the field is isotropic, by integrating the expression with the $\cos\theta^2$ we obtain the factor $1/3$.

1.2 Constancy of Intensity

We will now demonstrate a property of intensity that leads to a very counter-intuitive result, even if we experience it every day. The property is that, when the radiation is undisturbed (no absorption or scattering), then the specific intensity does not depend on distance. Suppose to have the usual bundle of rays crossing an area dA_1 with a solid angle $d\Omega_1$. Let's also assume that, at a distance R from the area dA_1, we have another area dA_2 such that it can intercept all rays emerging from dA_1. The solid angle of the photons passing through dA_2 is $d\Omega_2$. From the definition of solid angle we have $d\Omega_1 = \frac{dA_2}{R^2}$ and $d\Omega_2 = \frac{dA_1}{R^2}$. Since the photons are not absorbed or scattered, we have that, for the conservation of energy:

$$dE = I_{\nu_1}\, dA_1\, d\Omega_1 dt\, d\nu_1 = I_{\nu_2}\, dA_2\, d\Omega_2\, dt\, d\nu_2 \tag{1.10}$$

Since $\nu_1 = \nu_2$, then we have that $I_{\nu_1} = I_{\nu_2}$, i.e., the specific intensity does not depend on distance.

What does this mean exactly? Suppose to have an object that can be resolved, for example let's take the Moon, whose diameter has an angular size of about 0.5 degrees in the sky. The Moon has an intensity that we can experience directly with our own eyes. Imagine to move the Moon at twice its distance. What would we see? Will the Moon by dimmer? Since we have pushed the Moon at twice the distance, we will receive one fourth of the photons, according to the usual law of the square distance. However, at the same time more area of the Moon will be included within a certain solid angle which compensates exactly for the distance. Therefore even if the total number of photons has decreased by a factor 4, the number of photons received per unit solid angle will be the same. In other words, if you take a picture of the Moon, the correct exposure time *will not change* even if you have moved the Moon further away.

An everyday example might clarify this effect even further. When you take a picture of a sequence of objects that look all the same, say a few buildings along a street or a sequence of trees along a country road, you do not see objects far away underexposed and those in front overexposed. The are either all correctly or all incorrectly exposed. The same is with the eyes, we see the objects far away as having the same brightness (intensity) as those close-by. The only thing that changes is that those far away objects will have a smaller angular size and thus we will receive in total a smaller amount of photons, even if the amount of photons per unit solid angle is exactly the same.

A final example: if you go to Mars, or to Pluto for that matters, you take a picture of the Sun with a CCD camera and then measure the value of a pixel you will get the same exact number, although the total number of pixels that have detected photons has decreased the further away you go (because the Sun has decreased its angular size).

What about stars? Stars are not resolved objects so they do not have an intensity (i.e., no surface brightness). Strictly speaking they do appear as little dots on a camera (the Airy disk due to diffraction, plus aberrations). However, the Airy disk area does not change with distance since its radius is given by the diffraction limit (for a perfect optical system). Therefore if you move a star further away or closer you see only a change of flux, and therefore a star far away appears dimmer than close by and, if you take a picture, you need to vary your exposure time to compensate for the distance.

1.3 Transfer theory

In astrophysical settings photons reach us in two steps. First they are created by some radiative process (e.g., synchrotron, thermal emission, etc.). Then they propagate through space where they might interact with matter in several ways. They can be absorbed, scattered or more photons might be added by local sources. For example, if a star emits thermal radiation, the photons might be partially absorbed and scattered by dust and some additional infrared emission might be added from the dust itself. Absorption, scattering and emission and therefore three fundamental steps that modify the light we receive.

In this context, transfer theory provides a mathematical framework to model these interactions, under the assumptions that light moves across space in bundles of rays. The main purpose of transfer theory is to specify how specific intensity, calculated for a specific radiative process, is affected by these three aforementioned phenomena: absorption, scattering and emission.

When coupled with the equations of hydrostatic equilibrium and those of thermal and chemical balance, transfer theory provides a way to calculate several physical properties of a body like the cooling and heating rate and the radiation field throughout the body. When dust, molecules and/or atomic species are present in the medium or in the emitting body, radiative transfer provides also a way to calculate the emission/absorption characteristics of these bodies.

1.3.1 Emission

We have seen in the previous sections that, if a bundle of rays propagates in free space, the specific intensity remains constant. We will now drop the assumption of free space and introduce the possibility of a local contribution to the emission. Let's assume a source with specific intensity I_ν emits photons that cross a medium where some local sources are present. Let's assume that a bundle of rays cross an infinitesimal area dA and move by a length ds along the direction of propagation. If some sources are present in the volume $dV = dA\,ds$ emitting a certain amount of energy in a certain direction, then the local contribution to the energy present in the volume dV will be:

$$dE = j_\nu \, dV \, d\Omega \, dt \, d\nu \tag{1.11}$$

where the symbols have the usual meaning and j_ν is the specific emission coefficient. This specific emission coefficient has therefore dimensions of energy per unit volume, unit solid angle, unit time and unit frequency.

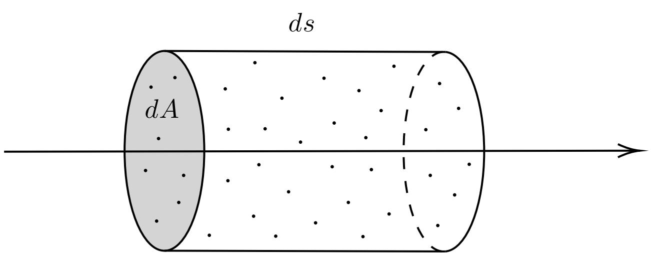

Figure 1.3: Source function

The solid angle as usual has the meaning of giving the direction of the emission. If the emission is isotropic then the specific emission coefficient takes the form:

$$j_\nu = \frac{1}{4\pi} P_\nu \tag{1.12}$$

where P_ν is the power emitted per unit volue per unit frequency.

Since the specific emission coefficient has the same units as the specific intensity times a length, the amount of specific intensity introduced by local emitting sources producing rays travelling a distance ds can be calculated as:

$$dI_\nu = j_\nu \, ds \tag{1.13}$$

Solving this simple equation gives:

$$I_\nu = I_{\nu,0} + \int_0^S j_\nu \, ds \tag{1.14}$$

which states that the specific intensity is equal to its initial value plus the contribution due to the emission coefficient integrated along the line of sight.

1.3.2 Absorption

If the bundle of rays are crossing a medium where local absorbers are present, then the specific intensity will decrease due to the absorption properties of the medium. Let's consider the same volume $dV = dA \, ds$ as we did for the emission, but this time let's fill it with N absorbers each having a cross section σ_ν. The cross section has units of an area $[L^2]$ and it is linked to the probability of an absorption event between the absorber and a ray belonging to the ray bundle. The number density of absorbers in the volume is n (units of $[L^{-3}]$) and therefore we can define an absorption coefficient:

$$\alpha_\nu = n \, \sigma_\nu \tag{1.15}$$

which has units of $[L^{-1}]$. Since the cross section σ_ν is related to the probability of absorption of a ray by one single absorber, the absorption coefficient α_ν is related to the probability of an absorption event by *any* of the absorbers present in the volume crossed by the bundle of rays. The variation of specific intensity will thus be proportional to: 1. the coefficient α_ν; 2. the initial unabsorbed specific intensity I_ν; 3. the length ds crossed by the rays:

$$dI_\nu = -\alpha_\nu \, I_\nu \, ds \tag{1.16}$$

Solving this equation one obtains:

$$I_\nu(s) = I_\nu(s_0) e^{-\int_{s_0}^{s} \alpha_n u(s') \, ds'} \tag{1.17}$$

This result simply states that the specific intensity decreases exponentially with the integral of the absorption coefficient integrated along the line of sight. A useful quantity to describe the decrease of the specific intensity along a certain trajectory is that of the *optical depth*. In differential form it can be defined as:

$$d\tau_\nu = \alpha_\nu \, ds = n\sigma_\nu \, ds \tag{1.18}$$

If there is only absorption involved, the expression for the specific intensity Eq.(1.17) can be rewritten as:

$$I_\nu(s) = I_\nu(s_0)e^{-\tau_\nu} \tag{1.19}$$

It follows that for $\tau_\nu = 1$ we have a decrease of $1/e$ in specific intensity. This value is usually taken as a reference to distinguish an optically thin medium ($\tau_\nu < 1$) from an optically thick one ($\tau_\nu > 1$). This distinction needs to be taken with a grain of salt since the optical depth is a continuous function and therefore there is no effective physical difference between values of τ_ν slightly smaller and slightly larger than 1[1]. As an illustrative example, if we take the optical depth of a mass of water with depth of 2 m, then the optical depth is roughly $\tau_\nu \approx 1$ for ν in the range of optical wavelengths. Indeed there is no difficulty in seeing the bottom of a swimming pool, although it does not appear as bright as is the case when the swimming pool is empty of water. By decreasing the thickness of the water layer to, say, 20 cm, the optical depth decreases by the same amount ($\tau_\nu \approx 0.1$) and indeed our experience suggests that we feel no decrease in brightness when looking at the bottom of a glass of water. If the optical depth is now increased to, say, $\tau_\nu \approx 10$, corresponding to a water layer of 200m, we have no chance of observing the bottom of the sea for example.

Another common way to describe the absorption of rays by a medium is by means of the mass absorption coefficient k_ν with units of $[L^3/M]$. This mass absorption coefficient is then multiplied by the density ρ of the absorbing material to obtain:

$$dI_\nu = -k_\nu \, \rho \, I_\nu \, ds \tag{1.20}$$

This way of writing the absorption is convenient if we know for example the type of particle responsible for the absorption. By knowing the particle mass m then we can immediately obtain the cross section of the particle $\sigma_n u = k_\nu \, m$. Furthermore, if the medium is composed by many particle species, one can write:

$$dI_\nu = -\sum_i k_{\nu,i} \, \rho_i \, I_\nu \, ds \tag{1.21}$$

[1]Some authors indeed define an optically thin and thick medium as $\tau_\nu \ll 1$ and $\tau_\nu \gg 1$, respectively.

1.3.3 Scattering

Scattering of radiation is a complex process that involves the interaction of a bundle of rays with a scattering object that deviates the direction of the rays. In some scattering processes (like in direct and inverse Compton scattering) the energy of the rays can be altered, whereas in some others (like in Thomson scattering) the energy is unaltered and the rays change direction according to a certain probability distribution.

In the context of radiative transport, scattering can be associated with the absorption coefficient α_ν by defining a scattering coefficient ζ_ν with same dimensions as the absorption coefficient. By analogy with the absorption, we can also define a mass scattering coefficient $s_\nu = \zeta_\nu \, \rho$. To determine the change of specific intensity due to scattering we need to consider two cases. The first case is similar to what we have obtained by treating absorption, in the sense that a bundle of rays coming from a direction $\vec{n}$ about a solid angle $d\Omega$ will change direction and move out of sight in a new direction $\vec{n}'$ about a solid angle $d\Omega'$:

$$dI_\nu = -\zeta_\nu \, I_\nu = -s_\nu \, \rho \, I_\nu \, ds \tag{1.22}$$

The second case is instead more complicated and deals with rays scattered *from* an arbitrary direction $\vec{n}'$ about $d\Omega'$ into the direction $\vec{n}$ about $d\Omega$. In other words, these are rays that are scattered *into* the line of sight of the observer even if their initial propagation direction (before the scattering) was not pointing to the observer. In this case one needs to consider the so-called phase function, which is the angular distribution of the specific intensity scattered by a particle at a given wavelength. We will encounter some famous examples of phase functions, for example the peanut shaped phase function of Thomson scattering. The phase function is nothing more than a probability density function, telling how likely it is for a photon coming from a solid angle Ω' to be scattered in a different specific solid angle Ω. The phase function does not represent the probability of a scattering event, since this information is encoded in the scattering coefficient ζ_s (or s_ν).

Since the phase function is a probability density function, then it must satisfy the condition:

$$\int_\Omega Pr_\nu(\Omega', \Omega) d\Omega' = 4\pi \tag{1.23}$$

where the symbol Pr represents a probability. The expression for the variation of the specific intensity becomes in this case:

$$dI_\nu = \frac{s_\nu \rho}{4\pi} \int I_\nu(\Omega')\, Pr_\nu(\vec{n}, \vec{n}') d\Omega' ds \tag{1.24}$$

where it is important to note that now I_ν depends on the solid angle and must be kept inside the integral.

1.3.4 Equation of Radiative Transport

The equation of radiative transport simply states that a bundle of rays travelling through a medium loses energy by absorption, gains energy by emission of local sources and redistributes energy as a consequence of scattering processes. We begin by defining the equation of radiative transport in the simplest form by ignoring first the scattering process. The reason for this choice is that, as can be seen in Eq.(1.24), scattering involves both a differentiation and an integration, so its calculation requires either numerical integration or approximations. We will see in detail what happens in this case, but to grasp the meaning of some fundamental concept we will begin with the simplified radiative transport equation.

By combining the decrement of I_ν when passing through a path of length ds and the contribution to I_ν from emitters, we obtain the simplified form of the so-called radiation transport equation:

$$\frac{dI_\nu}{ds} = -\alpha_\nu\, I_\nu + j_\nu \tag{1.25}$$

We will use this equation in the following in order to illustrate the meaning of some important quantities like the source function and the optical depth.

1.4 Source Function

We can rewrite Eq.(1.25) by defining the *source function* S_ν as the ratio between the emission and absorption coefficient:

$$\frac{dI_\nu}{ds} = -I_\nu + S_\nu \tag{1.26}$$

The source function is not only a useful quantity to rewrite the equation of radiative transport in a simpler way, but it has also a profound physical meaning. The source function, having the dimensions of a specific intensity, can be thought of as the local input of radiation. For example when the optical depth $\tau_\nu \gg 1$ then $I_\nu \approx S_\nu$ which means that matter far from the location of interest is giving a negligibly small contribution to the specific intensity. In other words, the source function is the value approached by the specific intensity given sufficient optical depth.

The formal solution for the equation of transport then becomes:

$$I_\nu(\tau_\nu) = I_\nu(s_0)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-}(\tau_\nu - \tau'_\nu)\, S_\nu(\tau'_\nu)\, d\tau'_\nu \; . \tag{1.27}$$

If the source function is constant, then:

$$I_\nu(\tau_\nu) = I_{\nu,0}\, e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \tag{1.28}$$

which has asymptotic solutions:

$$\tau_\nu \gg 1 \rightarrow I_\nu \approx S_\nu \tag{1.29}$$

$$\tau_\nu \ll 1 \rightarrow I_\nu \approx I_{\nu,0} + S_\nu\, \tau_\nu \tag{1.30}$$

1.4.1 The Meaning of the Source Function

Let's illustrate the meaning of the source function with an astrophysical example. An astrophysical source has intrinsic specific brightness $I_{\nu,0}$ and emits a number of rays that cross a medium with optical depth τ_ν. After the usual absorption and emission processes, the modified specific brightness reaches the observer that detects the photons with a receiver.

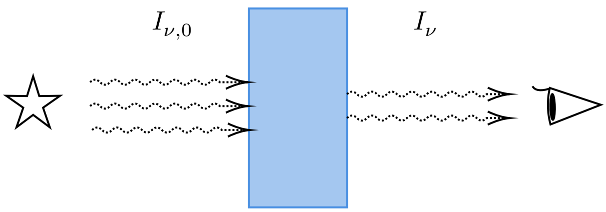

Figure 1.4: Source function.

For the medium, let's choose a uniform slab of gas whose optical thickness is $\ll 1$, for example $\tau_\nu = 0.02$ for $\nu = 30\,\mathrm{GHz}$. From the equation of transport Eq.(1.27) we see that the specific brightness observed at 30 GHz will be about 98% of the intrinsic specific brightness $I_{\nu,0}$ (due to the absorption process) plus a small contribution from the slab of gas itself. The source function will be equal to the specific intensity corresponding to a specific emission process. For example, as we will see in later chapters, if the slab of gas is the Earth atmosphere, then, under the zero-th order approximation that it is in thermal equilibrium, the source function will represent blackbody emission, so that $S_\nu(1-e^{-\tau_\nu})$ is the "background" contribution of the sky observed by the receiver[2]. The radiation that emerges after crossing the slab of gas is therefore equal to the incident radiation attenuated by the optical depth plus the sum of each layer of the slab emission attenuated by the optical depth from that point to the receiver.

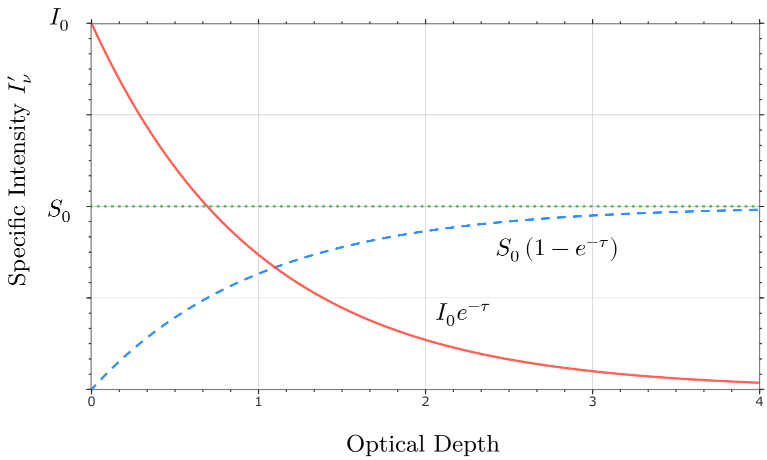

Figure 1.5: Source function

1.5 Mean Free Path

The mean free path is the average distance travelled by a photon before being either scattered or absorbed. When a bundle of rays is crossing a medium with optical depth τ_ν then the probability of crossing the medium up to a distance that corresponds to that optical depth τ_ν is given by $e^{-\tau_\nu}$. Therefore the mean optical depth can be found by integrating over all optical depths and averaging over the probabilities:

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu\, e^{\tau_\nu}\, d\tau_\nu = 1\ . \tag{1.31}$$

[2]The atmosphere of the Earth is not in thermal equilibrium, but in local thermodynamic equilibrium and it has a density and temperature gradient.

The fact that $\langle \tau_\nu \rangle = 1$ means that, *on average*, a photon travels for a distance such that the optical depth at the time of scattering/absorption is equal to one. Since the optical depth is a dimensionless number and it is defined as the absorption coefficient times the path length, we can define an average distance travelled by the photon before the scattering/absorption process:

$$\ell_\nu = \frac{\langle \tau_\nu \rangle}{\alpha_\nu} = \frac{1}{\alpha_\nu} \; . \tag{1.32}$$

As an example, suppose to be in presence of a spherical source with radius R and optical depth τ_ν. The mean free path is then:

$$\ell_\nu = \frac{1}{\alpha_\nu} = \frac{R}{n\,\sigma_\nu\,R} = \frac{R}{\tau_\nu} \tag{1.33}$$

which tells us that, for a fixed optical depth, the mean free path is proportional to the size of the object being crossed. This is intuitively simple to understand, since for a fixed optical depth, a larger source will be more "diluted" than a smaller one, so that the photons will travel a longer distance before being absorbed or scattered.

1.5.1 Random Walk

Closely related to the concept of mean free path is that of *random walk*. A random walk is a general type of random process involving a series of steps in random directions, where the choice of the random direction is determined according to some probability rule. Random walks take place in many different settings and according to many different rules. The simplest form of random walk occurs in one dimension, for example by taking the line of integer numbers and moving by one step along the positive or negative direction according to a probability rule. This can be controlled by flipping a fair coin and assigning a movement of $+1$ or -1 for head or tail. By repeating this process N times, the current location spreads out to larger and larger numbers (both positive and negative) and the probability of being on a specific number follows a Gaussian probability density function centered at the initial starting location. The interval corresponding to one standard deviation can be found by taking $\sqrt{N}$, so that if we flip the coin 100 times, there is a 68% chance of having the final location between $\pm\sqrt{N} = \pm 10$. When going from 1 dimension to 3 dimensions (or to *any* dimension), this general rule still applies.

We can apply this intuitive explanation of a random walk to the case of photons being *isotropically* scattered many times by particles. Each interaction occurs, on average, after the photon has crossed a length equal to the mean free path. The direction the photon will have after each interaction is randomly chosen according to a probability rule associated to the type of interaction occurring during the scattering process. How far would have the photons travelled after N interactions? Let's call $\vec{R}$ the displacement resulting from the sum of N displacements $\vec{r}_i$ with $i = 1, 2, ... N$:

$$\vec{R} = \vec{r}_1 + \vec{r}_2 + ... + \vec{r}_N \ . \tag{1.34}$$

If we average $\vec{R}$ we get exactly zero, which is expected since this would give the average of our Gaussian distribution function (for N large enough).

However, if we calculate the mean squared average distance (for reasons that will become clear below) we obtain a non-zero result:

$$\langle \vec{R}^2 \rangle = \langle \vec{r}_1^2 \rangle + \langle \vec{r}_2^2 \rangle + ... + \langle \vec{r}_N^2 \rangle + 2\langle \vec{r}_1 \vec{r}_2 \rangle + 2\langle \vec{r}_1 \vec{r}_3 \rangle + ... \tag{1.35}$$

The cross products are all equal zero, since these are averages over all directions (isotropic scattering) and so the end result is:

$$\langle \vec{R}^2 \rangle = N\ell^2 \tag{1.36}$$

where ℓ is the mean free path of our photon. What does $\langle \vec{R}^2 \rangle$ represent physically, and why was it necessary to calculate it? Let's go back to our example of the 1-dimensional random walk with a fair coin flip. In our example we used $N = 100$ but intuitively it is really easy to understand that if we run a sequence of coin flips for $N = 100$, it is very difficult to obtain exactly 50 head and 50 tails. The number of head and tails will be close but uneven, with a slight excess of one of the two, for example 47 heads and 53 tails. If we take a much larger number, say $N = 10^6$, the situation will be similar, it is even more unlikely to obtain exactly 500,000 heads and 500,000 tails. This difference between the number of heads and tails in this kind of experiment grows with the number of coin flips. It grows precisely as the square root of the number of flips. Therefore, going back to the case of photons, the root mean square displacement $\sqrt{\langle R \rangle} = \sqrt{N}\ell$ will be the net average displacement of the photon. If we go back to Eq.(1.33) then we can link together the mean free path, the total displacement and the optical depth of the source. Indeed we obtain the remarkable fact that the optical depth is a measure of the number of scattering events that a photon experiences, on average, when crossing the medium:

$$\tau = \sqrt{N} \ . \tag{1.37}$$

This result is, however, valid only for an optically thick medium. In case of an optically thin material, the photon will have a mean free path larger than the size occupied by the medium itself, and therefore the validity of the former equation breaks down and $\tau = N$ is a more appropriate expression for the number of interactions.

1.6 Examples

1.6.1 HL Tauris

T-Tauris stars are a type of variable protostars that are in the process of contracting before triggering nuclear fusion and starting their lives on the main sequence. These objects are surrounded by very large protoplanetary disks of gas and dust and sometimes also young planets. T-Tauris stars are usually younger than 10 Myr and therefore provide a great laboratory to understand stellar and planet formation.

A well known example of a T-Tauris star is the system called HL Tauri, which is at about 450 light years distance and has a very large protoplanetary disk showing dark bands probably created by newly formed planets. The Atacama Large Millimeter/submillimeter Array located in Chile is an astronomical radio interferometer that has recently observed in great detail HL Tauri. In Fig. 1.6.1 we show an ALMA image taken in 2014, where the genesis of a new star and planetary system is recorder with unprecedented detail and resolution.

The center of the image shows the glow of the protostar in the process of accreting gas from the surrounding disk. The protoplanetary disk extends for about 200 AU and has a mass of the order of $0.1\,M_{\odot}$. The structure of the disk varies with the distance from the protostar, with a turbulent region of hot gas at high temperature (up to 1,500 K) and then a colder region further out where complex molecules can form together with grains of dust and ice (see Fig. 1.6.1).

Radiative transport plays a crucial role in order to understand the complex structure of these type of disks. Numerical codes are usually used to study the physics of these objects, where the equations of hydrostatic equilibrium are coupled with those of radiative transport and then to the equations that describe the chemistry and thermal balance of the disk. To solve the radiative transport equation in this case, the dust and gas opacities are first defined along with an input radiation field (that depends on the type of local emission, for example thermal emission). The equations are then solved numerically and the results are used to fit

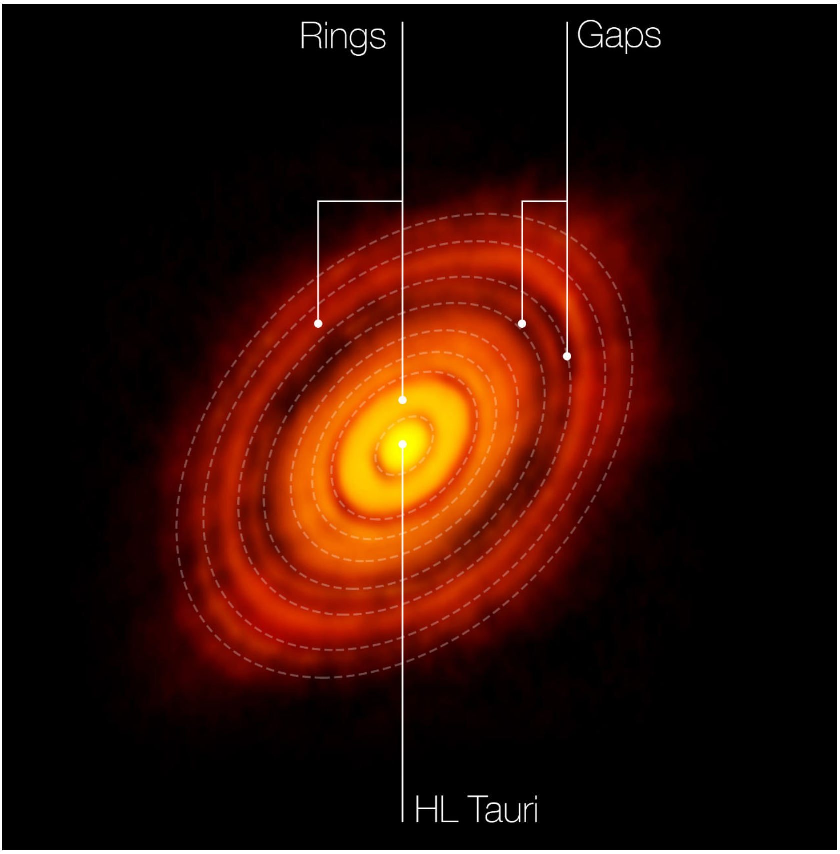

Figure 1.6: Source function.

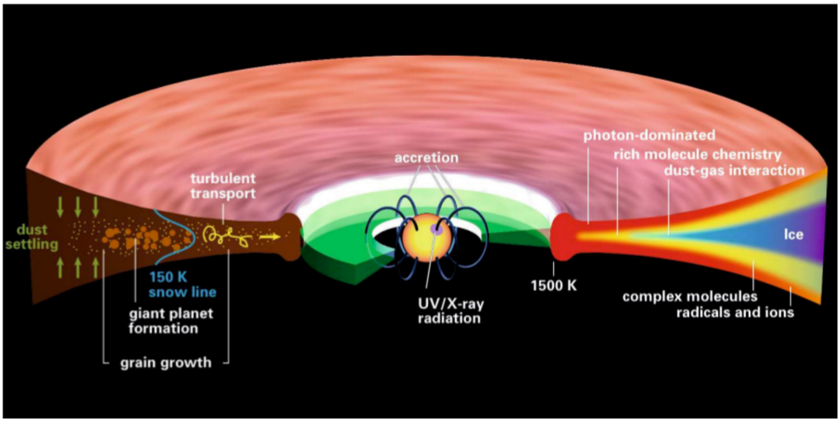

Figure 1.7: Source function.

a model to the spectral energy distribution observed with the telescopes, determine the strength of the emission and/or atomic and molecular absorption lines and the geometric parameters of the system like the disk mass, inclination etc.

1.6.2 Planetary Nebulae

Planetary nebulae represent the final stages of stellar evolution of low mass stars. The Sun will meet its fate by blowing away its outer gas layers and becoming a white dwarf surrounded by a planetary nebula. When looking the the images of such objects, it is common to see a layer of gas with a central hole through which it is sometimes possible to detect the white dwarf itself. Why do we see a central hole since these objects are supposed to be roughly spherically symmetric? The main reason is that the optical depth varies with the line of sight. When we look through the center of the nebula, the path s crossed by the rays is much smaller than the path crossed by the rays emitted on the border of the nebula. Therefore, even if the absorption coefficient is roughly the same everywhere in the nebula, $\tau_\nu \ll 1$ towards the center and it becomes optically thick towards the outer edges because s increases. This means that the photons coming from central part of the nebula are almost undisturbed (number of interactions $N \sim \tau \ll 1$).

Figure 1.8: Planetary nebula Helix.

1.6.3 Why the Sky is Blue and the Sun is Red at Dusk

The Rayleigh scattering occurring in the Earth atmosphere has a strong dependence of the photon frequency. Indeed the cross-section of the scattered light scales as $\sigma_\nu \propto \nu^4$. The scattering therefore is very large for blue light which has a much larger random walk in the sky. The mean free path for blue light:

$$\ell_\nu = \frac{1}{\sigma_\nu n} \tag{1.38}$$

is small and the number of scatterings N is large and the optical depth for blue is larger than for red. However, the optical depth for blue light is still smaller than one, so only a small part of the radiation is scattered away. Red light has a much larger mean free path, N is small and the region of the sky affected by the scattering is smaller, thus the sky has a blue hue during the day. At sunset, the amount of atmosphere crossed by sunlight increases by a large amount, so that almost all blue radiation is scattered away ($\tau_{\rm blue} > 1$) whereas the red radiation is still in the optically thin regime ($\tau_{\rm red} < 1$) and thus we see mostly red light for both the sun and the region of the sky close to the horizon (for which $\tau_{\rm blue} > 1$).

1.7 Einstein Coefficients I

When discussing the emission coefficient we have focused so far only on the spontaneous emission of photons in a medium. However, as A. Einstein first noted, if one considers two energy states for an atom, there are three possible outcomes. The first is the absorption of a photon that causes the electron in the atom to transition from one energy level n_1 to the next n_2 (or $|1\rangle$ and $|2\rangle$ in bra-ket notation). The second is the spontaneous emission that brings an excited atom from n_2 to n_1. And the third possibility is stimulated emission, where an incident photon on an excited atom in n_2 induces the emission of a second photon that brings the atom from n_2 to n_1.

A. Einstein defined three coefficients to describe the three types of transitions: A_{21}, B_{12} and B_{21} for spontaneous emission, absorption and stimulated emission, respectively. The subscripts 1 and 2 identify the initial and final energy state of the atom. The three coefficients, as we will see below, do not have the same dimensions, since they need to be combined with other quantities relevant for the specific process in order to give a transition probability. The emission coefficient A_{12} has units of a frequency s^{-1} ($[T^{-1}]$) and it can be interpreted as the transition probability for spontaneous emission per unit time. The coefficients B_{12} and B_{21} have both units of $\mathrm{erg}^{-1}\,\mathrm{cm}^3\,\mathrm{s}^{-2}$ and so $[M\,L^{-1}]$. These coefficients represent the probability per unit time per unit spectral energy density of the radiation field that an electron in state n_2 will decay to state n_1. The fact that both absorption and stimulated emission have the same units is due to the fact that one can consider stimulated emission as *negative absorption* since the process is identical to the absorption of a photon, with the difference that instead of having the incident photon disappear after the absorption event, there are now two photons with the same energy.

The Einstein coefficients are useful because they are atomic properties that do not depend on specific thermodynamic conditions and therefore have a general validity. They will play an important role when we will need to calculate the emission coefficient (Eq. 1.11) in certain specific cases.

The populations n_2 and n_1 and the Einstein coefficients are related with the following three equations:

$$\frac{d\,n_1}{dt} = -B_{12}\,n_1\,\rho(\nu) \qquad \text{Absorption} \tag{1.39a}$$

$$\frac{d\,n_2}{dt} = -A_{21}\,n_2 \qquad \text{Spontaneous emission} \tag{1.39b}$$

$$\frac{d\,n_2}{dt} = -B_{21}\,n_2\,\rho(\nu) \qquad \text{Stimulated emission} \tag{1.39c}$$

where $\rho(\nu)$ represents the spectral energy density with units of energy per unit volume per unit frequency ($[M\,L^{-1}\,T^{-1}]$).