Shocks in Astrophysical Plasmas

Notes for lecture 11

Shocks

When an astrophysical plasma moves at supersonic speeds, like in the case of accretion flows on compact objects, the gas velocity might at some point reach a halt. For example, when the gas is accreted on the surface of a neutron star it has to transition from a state where the speed is a significant fraction of the speed of light to a state where the flow is subsonic before settling on the surface. The same happens when a blob or a shell of material is ejected in a supernova explosion or in other energetic phenomena and collides with a medium at a lower speed (or at rest).

In the reference frame of the moving plasma, the surface of the object being approached is seen as moving with the velocity of the plasma towards the plasma. When the plasma collides with the object we can assume that, since the velocities are so large, any change occurring in the plasma is adiabatic. Remember that the speed of sound is:

$$c_s = \sqrt{\frac{dP}{d\rho}} \tag{1}$$

where P is the plasma pressure and ρ its density. If the plasma is adiabatic, the equation of state is:

$$P = K\rho^{5/3} \tag{2}$$

with K a constant. As the plasma is compressed, its density increases and so does the sound speed since in the adiabatic case $c_s \propto \rho^{1/3}$.We can think about it as a fluid in a cylinder being compressed by a piston (see Figure 1).

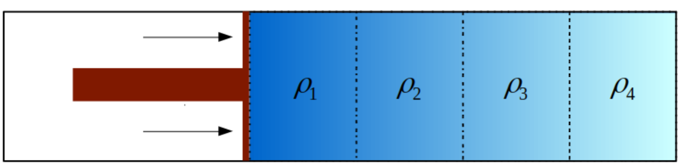

Figure 1. Piston model. The fluid is compressed by a piston (red) moving with velocity $\vec{v}$ (black arrows).

The fluid is initially at rest and we consider only one dimension (say the positive x-axis, along the longest side of the cylinder), so a plane-parallel setup. The speed of sound initially is given by $c_s = c_0 = const$. At time $t = 0$ we move the piston in the direction of the positive x-axis to compress the fluid and we assume that the piston has a constant acceleration a. The fluid closest to the piston increases its density. This perturbation propagates further away with a certain delay due to the finite speed of sound and the density gradient increases with time. This process continues until the density scale length $\rho/\left|\vec{\nabla}\rho\right|$ is of the order of the mean free path $\lambda_{\rm mfp}$. Indeed when this happens, the transition region between the perturbed and unperturbed gas is very narrow and viscous forces will dissipate the ordered kinetic energy and transform it into thermal motion.

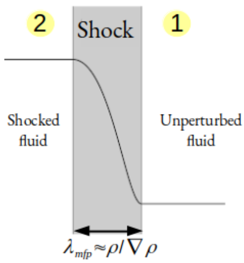

Figure 2. Shock formation. The shock is a thin layer of length $\rho/\left|\vec{\nabla}\rho\right|$. In the reference frame of the shock, the shock will see a gas in zone (1) with P_1, ρ_1 and v_1 emerge in zone (2) with P_2, ρ_2 and v_2, where $P_2 > P_1, \rho_2 > \rho_1$ and $v_2 < v_1$.

This happens because the ratio between the viscous force F_v and pressure gradient becomes of order unity or larger. The viscous force is given by the (supersonic) speed of particles $v > c_s$ and can be multiplied to the mean free path $\lambda_{\rm mfp}$ to get a force per unit mass[1]. The pressure force instead is given by the speed of sound c_s times the scale length of pressure density variations, which is L.

$$\frac{\text{viscous force}}{\text{pressure force}} \sim \frac{v}{c_s}\frac{\lambda_{mfp}}{L} \tag{3}$$

When $L \sim \lambda_{\rm mfp}$, this ratio becomes significant and the viscosity starts to dominate in the transition region.

If we are in the rest frame of the shock, then we will see the gas moving from region 1 (see Figure 2) with pressure P_1, density ρ_1 and velocity v_1 and emerge with a larger pressure and density P_2, ρ_2 and smaller velocity v_2. It is possible to demonstrate that the ram pressure before the shock ρv_1^2 is dominant with respect to the thermal pressure. After the gas has been shocked, the thermal pressure becomes larger (or of the same order of) the ram pressure ρv_2^2. This means that the kinetic energy has been converted to thermal energy thanks to the presence of a shock. A shock wave in general:

- is a disturbance in a fluid that is driven by compression
- is a disturbance propagating faster than the local speed of sound
- produces an increase of entropy in the fluid and an irreversible change in the thermodynamic properties of the fluid

[1]remember that the force has units of $\text{g} \cdot \text{cm}^2/\text{s}$ in cgs units.

Jump Conditions

When the gas moves through the shock, the relationship between the states on both sides of the shock wave (Figure 2) in a one-dimensional flow are called *jump conditions*.

These jump conditions, or Rankine–Hugoniot conditions, can be derived by using three conservation laws (together with the equation of state of the fluid, e.g., Eq. 2):

- conservation of mass
- conservation of momentum
- conservation of energy.

Conservation of Mass

The conservation of mass in the case of fluids is called *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x\right) = 0 \tag{4}$$

Since the shock is already in a steady-state and we have no accumulation of mass at the interface with length λmfp, we can set the time variations equal to zero. Furthermore, since the thickness of the shock is tiny, $L \sim \lambda$mfp, we can set $dx \sim \lambda_{\mathrm{mfp}}$ and therefore:

$$\frac{\partial}{\partial x}\left(\rho v_x\right) = \rho v_x|_{x=dx/2} - \rho v_x|_{x=-dx/2} = 0 \tag{5}$$

from which we have:

$$\rho_1 v_1 = \rho_2 v_2. \tag{6}$$

The equation above is called *first jump condition first Rankine-Hugoniot relation*.

Conservation of Momentum

For the momentum equation we can use Euler equation (the equivalent of the 2nd Newton law for fluids):

$$\rho \frac{\partial v_x}{\partial t} + \rho v_x \cdot \frac{\partial}{\partial x} v_x = -\frac{\partial}{\partial x} P + \vec{f}. \tag{7}$$

In this situation there are no external forces acting on the fluid, so that $\vec{f} = 0$ and since we are in steady state we set all time derivatives to zero. We get:

$$-(\rho v_x v_x + P)|_{dx/2} + (\rho v_x v_x + P)|_{-dx/2} = 0 \tag{8}$$

and therefore we obtain:

$$\rho_1 v_1^2 + P_1 = \rho_2 v_2^2 + P_2 \tag{9}$$

The equation above is called *second jump condition* or *second Rankine-Hugoniot relation*.

Conservation of Energy

The conservation of energy equation can be written by ignoring heat losses/gains ($\dot{Q} = 0$, since we are assuming adiabatic conditions) and assuming no external forces. In this case a fluid element will have a kinetic energy density $\frac{1}{2}\rho v^2$ plus internal (e.g., thermal) energy density $\rho\epsilon$ where ϵ is the internal energy per unit mass. For a thermal fluid, the internal energy depends on temperature T and from the equipartition theorem we know that each degree of freedom has a mean energy of $\frac{1}{2}k_B T$, so that $\epsilon = \frac{3k_B T}{2\mu m_H}$, where μ is the molecular weight and m_H the hydrogen mass. Let's call the total energy density of a fluid element as $\xi = \frac{1}{2}\rho v^2 + \rho\epsilon$. The equation of energy can thus be written as:

$$\frac{\partial \xi}{\partial t} + \nabla \cdot [(\xi + P)\, \vec{v}] = 0 \tag{10}$$

Again, in steady state conditions we put time variations to zero and we get:

$$(\xi_1 + P_1)v_1 = (\xi_2 + P_2)v_2 \tag{11}$$

which is the *third jump condition* or the *third Rankine-Hugoniot relation.*

By coupling these three equations with the equation of state of the fluid, one rewrite these three equations as follows:

$$\text{Jump Conditions} = \begin{cases} \dfrac{\rho_2}{\rho_1} = \dfrac{\mathrm{v}_1}{\mathrm{v}_2} = \dfrac{(\gamma+1)\,\mathrm{M}_1^2}{(\gamma-1)\,\mathrm{M}_1^2+2} \\ \dfrac{\mathrm{P}_2}{\mathrm{P}_1} = \dfrac{2\gamma\mathrm{M}_1^2-(\gamma-1)}{\gamma+1} \\ \dfrac{\mathrm{T}_2}{\mathrm{T}_1} = \dfrac{\left[(\gamma-1)\,\mathrm{M}_1^2+2\right]\left[2\gamma\mathrm{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathrm{M}_1^2} \end{cases} \tag{12}$$

where $M_1 = v_1/c_{s,1}$ is the Mach number of the incoming flow. For strong shocks we have usually $M_1 \gg 1$ and therefore $\rho_2/\rho_1 = 4$ when $\gamma = 5/3$ (adiabatic condition). Similarly, for strong shocks we can derive the temperature of the shocked fluid as:

$$T_2 = \frac{3}{8k_B}\left(\frac{1}{2}\mu m_H v_1^2\right) \approx 2 \times 10^{12}\left(\frac{v_1}{c}\right)^2 K \tag{13}$$

again calculated for $\gamma = 5/3$ (here c is the speed of light as usual).