Accretion Disks and Viscosity

Notes on Lecture 10

Formation of an Accretion Disc

Circularization Radius

When the accreting gas has some angular momentum associated with it, then it forms a so called *accretion disc*. The accreting gas forms a disc around the central attracting object and a particular parcel of matter gradually spirals inward, thereby decreasing its negative gravitational potential energy. There is strong circumstantial evidence that this is the process by which energy is released in many astronomical systems. The first astronomical observations in the X-ray band (Giacconi et al. 1962) established the existence of compact X-ray sources. After the launch in 1970 of the satellite *Uhuru*, devoted completely to X-ray astronomy, it was possible to identify several compact X-ray sources with binary stellar systems. The most plausible model for such system is that the binary contains a neutron star (or black hole) and another "normal" gaseous star (called "donor star") of mass M_d, as sketched in Figure 1.

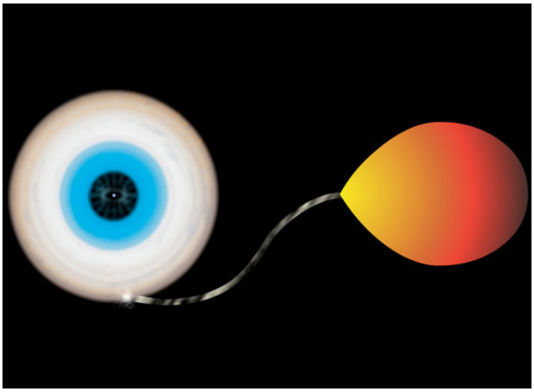

Figure 1. An X-ray binary, with a black hole accretor, an accretion disc and a low mass donor star ($M_d \lesssim 1\,M_\odot$). The donor star is filling the Roche lobe and transfers gas towards the accretor via the accretion disc.

Since the gas from the donor star has angular momentum due to the orbital motion of the donor around the center of mass of the binary, the gas flows towards the compact star by first forming an accretion disc. The gravitational energy lost is radiated in the form of X-rays.

Actually, the "accretor" can also be a white dwarf, although in that case the radiation emitted will not peak in X-rays but in the Optical/UV (for reason connected with the gravitational potential of the gas in the WD gravitational field, which is much weaker than in NS or BHs in the innermost regions of the accretion disc). Also the donor can indeed be different than a normal star, and be for example a brown or a white dwarf.

If the disc has a semi-thickness $h(r)$ and a radial coordinate r, then the thin disc model applies when $h(r)/r \ll 1$. This is true in many (but not all) X-ray binaries, and we will work under this hypothesis from now on. The thin accretion disc model was studied in detail in a famous paper[1] by Shakura & Sunyaev (1973, Astron. Astrophys., Vol. 24, p. 337 - 355).

When the gas leaves the donor star via the Lagrangian point L_1 (see Fig. 13.12 in S&T), it has a substantial angular momentum which is equal to the specific orbital angular momentum. To a good approximation, we can take the stream trajectory as the orbit of a test particle released from rest at L_1, and thus with a given angular momentum, falling in the gravitational field of the accretor alone. This would give an elliptical orbit lying in the binary plane: the presence of the secondary causes this to precess slowly. (The presence of the donor star changes indeed the effective gravitational potential from the exact $1/r$ dependence which is required for closed periodic orbits, as it would be if only the accretor were present). A continuous stream trying to follow this orbit will therefore intersect itself, resulting in dissipation of energy via shocks. On the other hand, the gas has little opportunity to rid itself of the angular momentum it has on leaving L_1, so it will tend to the orbit of lowest energy for a given angular momentum, i.e., a circular orbit. We thus expect the gas initially to orbit the primary in the binary plane at a radius R_{circ} such that the Kepler orbit at R_{circ} has the same specific angular momentum as the transferring gas has on passing through L_1. Thus the gas will have a circular velocity:

$$v_\phi = \left(\frac{GM}{R_{circ}}\right)^{1/2} \tag{1}$$

where we have used cylindrical coordinates (r, ϕ, z), with ϕ the azimuthal coordinate, G the universal gravitational constant and M the central body mass. At this point the gas flows in a thin ring and has to expand in the radial direction (both inwards and outwards) to create an accretion disc. The inertial forces of the gas will be given by $R_{circ} \cdot v_\phi$, which for a typical accretion disc are equal to approximately 10^{18} cm^2 s^{-1}. If the gas was collisionless, then the flow will proceed along the circularization ring and an accretion disc cannot form in any circumstance. The real gas flow instead will be an *hydrodynamic flow*, in the sense that collisions, and thus viscosity, has an important role. Now, to form a disc structure starting from a ring, part of the gas in the ring needs to move inward. But this is not sufficient, since the total angular momentum in the disc must be conserved because there is no external torque acting on the gas. Thus, to an inward motion there must be a corresponding outward gas motion. The gas spiraling inward/outward has less/more angular momentum than the original gas at the circularization radius. The net effect is that the total angular momentum is conserved and the original gas ring has now expanded into an extended accretion disc. To have this outward angular momentum transport, we need viscosity.

[1] this paper is one of the most cited in all fields of astronomy, with more than 5000 citations at the time of writing

The Role of Viscosity

Viscosity is the fluid internal resistance to flow and can be though as the equivalent of a "fluid friction". All fluids have a viscosity, except ideal fluids and superfluids. The accreting gas behaves as a viscous fluid, and viscosity has a fundamental role for at least two reasons:

- for the formation of the accretion disc itself (the outward/inward gas motion)
- for conversion of gravitational potential energy into radiation

However, viscosity is perhaps the most poorly understood property of accreting discs, despite several decades of intensive research in this field. The reason, which might appear surprising at first, is that we do not understand completely the origin of viscosity in the accretion disc gas. Why ? Can't we treat viscosity just as in normal fluids we know in everyday life (like water) ? The short answer is: no, we cannot.

The accreting gas flowing at the circularization radius will be laminar or turbulent according to the Reynold number:

$$\Re = \frac{inertial\ forces}{viscous\ forces} = \frac{R\, v_\phi}{\nu} \tag{2}$$

where ν is the viscosity of the gas. In lab experiments, the $\Re$ number gives turbulence when it is larger than about $\sim\ 10–1000$. Now, suppose the viscosity is given by the molecular scattering of particles, so that $\nu_{mol} = \lambda_{\rm mfp}\, c_s$, where $\lambda_{\rm mfp}$ is the mean free path of gas particles and c_s is the sound speed. The molecular scattering is for example responsible for the viscosity of air or water, just to give two familiar examples. It might appear natural therefore to extend this also to accretion discs.

If the accreting gas is sufficiently hot, then it will constitute a plasma with each plasma particle having a mean free path $\lambda_{\rm mfp} \simeq (n\sigma)^{-1} = \left(4\pi\, b_c^2\, n\right)^{-1}$. Here n is the number density of molecules (or particles in this case) and b_c is the "impact parameter" for plasma, which can be determined by considering the electrostatic interaction between the charged component of the plasma (for typical accretion discs the temperature is $\sim\ 10^4$ K so a plasma of electrons and protons is formed, since the gas is mainly composed by Hydrogen which is ionized above $\approx\ 6500$K). The impact parameter can be calculated by equating the electrostatic potential between Z charges of charge e and the thermal temperature of the plasma. Therefore $\left(Ze^2\right)/b_c \simeq\ (3/2)k_bT$. Numerically this gives $\lambda_{\rm mfp} \approx 6\times\ 10^4 \left(\frac{T^2}{n}\right)$ cm and $c_s \approx\ 10^4 T^{1/2}$ cm s^{-1}. Therefore the molecular viscosity can be calculated as $\nu_{mol} \simeq 6\times 10^8\, T^{5/2} n^{-1}$ cm^2 s^{-1}. For typical astrophysical accretion disc values ($n \approx\ 10^{15}\ cm^{-3}$, $T \approx\ 10^4 K$), the value of $\Re$ will be extremely high (typical values are $\Re > 10^{14}$). Therefore the molecular viscosity is completely irrelevant when compared to inertial forces, and therefore cannot be that important to determine the formation of an accretion disc (the situation will be almost identical to the collisionless gas case). To fix ideas, note that disturbances are propagated by viscous diffusion over a distance l on a time scale of order:

$$t_{visc} \sim\ l^2/\nu \tag{3}$$

where ν is the molecular viscosity. This timescale is about $\sim 10^8$ years for typical accretion disc sizes $l \sim 10^{10}$ cm and $\nu = 10^4$ cm^2 s^{-1}. This is orders of magnitude too long for the time variability seen in compact object accretion discs.

Since $\mathfrak{R}$ is very large for molecular viscosity, we might conjecture that the gas flow in accretion discs is also turbulent, although there is, as yet, no proof that this is so. If this is the case, the flow will be characterized by the size λ_{turb} and turnover velocity v_{turb} of the largest turbulent eddies. Since the turbulent motion is completely chaotic about the mean gas velocity, our simple viscosity calculations apply: there is a turbulent viscosity $\nu_{turb} = \lambda_{turb} v_{turb}$. Although this result has a neat appearance, it is here that our troubles with viscosity really begin. Turbulence is one of the major uncharted areas of classical physics and we do not understand the onset of turbulence, still less the physical mechanisms involved and how they determine the length-scale, λ_{turn}, and turnover velocity v_{turn}. The most we can do with present knowledge is to place plausible limits on these two parameters. First, the typical size of the largest turbulent eddies cannot exceed the disc thickness, so $\lambda_{turn} < H$, where we have used the disc scale height H. Second, it is unlikely that the turnover velocity is supersonic, for, in this case, the turbulent motions would probably be thermalized by shocks. Thus, we can write:

$$\nu = \alpha c_s H \tag{4}$$

and expect $\alpha < 1$. This is the so-called α-prescription of think accretion discs, and all our ignorance on the viscosity ν has just been isolated in α. However, apart from the rather obvious expectation $\alpha < 1$ we have gained nothing so far. Nonetheless, the α-prescription has proved a useful parametrization of our ignorance and has encouraged a semi-empirical approach to the viscosity problem, which seeks to estimate the magnitude of α by a comparison of theory and observation. There is, for example, some reason to believe that $\alpha \sim 0.1$ in accretion discs in accreting white dwarfs, at least some of the time.

Rayleigh Stability

There is however still a fundamental problem in this whole picture, in the sense that even if we have quite reasonably assumed that turbulence plays a role in determining the viscosity of the accretion disc, we still have not identified any energy source to *sustain* this turbulence. The most obvious solution to this problem is to invoke a hydrodynamic instability. However, the angular velocity Ω of the gas in the accretion disc scales as $r^{-3/2}$ (i.e., Keplerian motion), and therefore the gas will obey the *Rayleigh stability criterion*:

$$\frac{d\left(r^2\Omega\right)^2}{dr} > 0 \tag{5}$$

The above simple criterion is derived and explained in the Appendix of these notes.

Any realistic accretion disc (in the thin accreton disc case for example $\Omega \propto r^{-3/2}$) is therefore linearly stable to pure hydrodynamic perturbations: they satisfy the classical Rayleigh criterion for axisymmetric perturbation by a safe margin and they are also stable to non-axisymmetric disturbances. Therefore even if the shear in the mean flow generates turbulence with the desired transport of angular momentum outward, the Reynolds turbulent stress will inevitably decay (because it is Rayleigh stable) unless some other effect feeds energy back into the turbulence.

Magneto-Rotational Instability (MRI)

A mechanism that currently looks very promising in that it seems to satisfy all the consistency requirements mentioned above and predicts a viscosity of the right magnitude and sign is magneto-hydrodynamic (MHD) turbulence. MHD turbulence differs from the hydrodynamic turbulence described above in the sense that magnetic fields have now an effect on the plasma flow and the Rayleigh stability criterion outlined above does not apply anymore. Now, it can be demonstrated that with a magnetic field in the plasma, the stability is guaranteed only if:

$$\frac{d\Omega^2}{dr} > 0 \tag{6}$$

Therefore for a Keplerian disc with $\Omega \propto r^{-3/2}$, the stability criterion is not satisfied and the disc becomes unstable to axisymmetric perturbations. This idea of MHD turbulence is not new, but has received fresh impetus in the context of accretion discs through the recent rediscovery by Balbus and Hawley (1991) of a weak field instability originally discussed by Velikhov (1959) and Chandrasekhar (1960, 1961). The addition of a magnetic field implies that even though the perturbations are still axisymmetric, the angular momentum of each fluid element is no longer conserved because the magnetic field can apply stresses which move it about. Under these circumstances, instability to axisymmetric perturbations occurs when the angular velocity decreases outwards.

In an ideal plasma (i.e. a magnetized, perfectly conducting fluid), the action of the magnetic field is to link neighboring fluid parcels that lie along a common field line. One property of ideal plasmas is that magnetic field lines are frozen in the fluid parcels, so that the motion of the fluid carries the magnetic field along with it. This lends itself to a beautifully simple physical picture of the magnetic forces; the fluid parcels can be thought of a beads tied together on a string. If for some reason fluid parcels start to diverge, the tension in the magnetic "string" acts to bring the connected fluid elements back together (see Figure 2).

Intuitively, we can just think of the magnetic force acting on the fluid as springs tying neighboring fluid elements together! Normally, we think of a spring as a restoring force, which tends to preserve a system's stability. However, if the fluid motion occurs in a differentially rotating frame, it turns out that this restoring force can actually lead to the system destabilizing. Imagine that the red particle in Figure 2 is at a radius $r + \epsilon$ with respect to the blue particle, which is at a radius r. Here ϵ represents an infinitesimal displacement along the radial accretion disc coordinate. The red particle has therefore a slightly larger angular momentum than the blue particle and they start to rotate differentially on different Keplerian orbits. The blue particle will rotate faster than the red particle ($\Omega \propto r^{-3/2}$) and therefore the "magnetic spring", that connects the two particles, will pull the red particle. The pulled red particle will therefore increase its angular momentum, and move further outward. The blue particle will instead be slowed down by the red particle, and will move further inward. As the red and blue particles move outward and inward, the situation

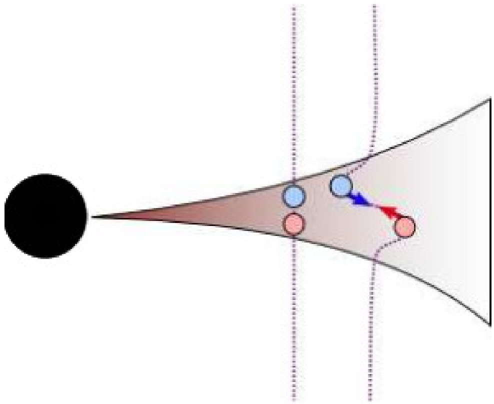

Figure 2. Motion of the fluid parcels (red and blue circles) carries the magnetic field lines (purple line) along with them. The resultant magnetic forces are shown (red and blue arrows) which arise due to tension in the field lines.

becomes worse and worse and the process keeps going on (see Figure 3). This is the origin of the MRI. Now it is clear how an accretion disc can develop from the simple circular ring of material we started with.

A schematic summary of the MRI goes as follows:

- a) Magnetically connected fluid elements have some initial displacement
- b) Differential rotation increases the displacement, and magnetic tension causes the inner parcel to slow down, and the outer parcel to speed up
- c) This transfer of angular momentum causes the inner parcel to migrate inwards and the outer parcel to be pushed outwards.
- d) Repeat from step a, but now with a larger displacement.

However, if the magnetic field is too strong (i.e. the tension in the spring is too strong), the feedback cycle will not run. The tension will instead cause the displacement between fluid parcels to oscillate rather than grow in step b).

Accretion disc structure

An accretion disc can conveniently be divided into three distinct regions, depending on r. These regions are:

- 1. An outer region, at large r, in which gas pressure dominates radiation pressure and in which the opacity is controlled by free-free absorption;
- 2. A middle region, at smaller r, in which gas pressure dominates radiation pressure but the opacity is mainly due to electron scattering; and

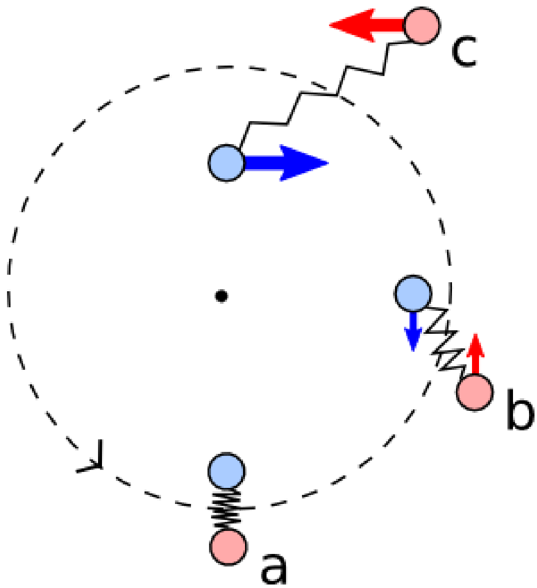

Figure 3. Growth of the MRI due to differential rotation

- 3. An inner region, at very small r, in which radiation pressure dominates gas pressure and again, scattering dominates absorption in the opacity. (For some choices of h, the inner and middle regions may not exist at all.)

Typically, the transition from the outer to middle region occurs at a radius r_{om} when $\bar{k}_{ff} \simeq \bar{k}_{es}$,which occurs at

$$\frac{2r_{om}}{r_s} = 4 \times 10^3 \left(\frac{\dot{M}_{17}}{M/M_\odot} \right)^{2/3} \tag{7}$$

Here $r_s = 2GM/c^2$ is the Schwarzschild radius, $\dot{M}_{17}$ is in units of 10^{17} g s^{-1}. The transition from the middle to the inner region (point r_{mi}) instead can be found when $P_{gas} \simeq P_{rad}$:

$$\frac{2r_{mi}}{r_s} = 80\alpha^{2/21} \left(\frac{M}{M_\odot} \right)^{-2/3} \dot{M}_{17}^{16/21} \tag{8}$$

Now, the gas flows in the accretion disc because of some form of turbulent viscosity (like the MRI sketched above), and this process heats up the disc from the outer regions towards the innermost part. The viscosity generates entropy (heat) which is released as thermal radiation. The integrated flux emitted from the top and bottom faces of the disk at r is:

$$F(r) = h(r)\dot{Q} = \frac{3\dot{M}}{8\pi r^2} \frac{GM}{r} \left[1 - \left(\frac{r_I}{r} \right)^{1/2} \right] \tag{9}$$

with $\dot{Q}$ the heat production rate, r_I the innermost disc radius and h the semi-thickness of the disc at radius r. The curious reader can find the derivation of this equation in §14.5 of the textbook S&T. Here we need just to know that this equation is found when assuming that no heat is stored in the disc but it is totally radiated away, that the disc has a shear flow and that mass and angular momentum are conserved.

To summarize: the particle leaves the donor at the lagrangian point L_1 and enters the gravitational potential well of the accretor. The gravitational potential energy is converted in kinetic energy of the particle that rotates around the circularization radius. The kinetic energy is then converted into heat by viscosity, which is also responsible for the formation of the accretion disc. As the disc forms, an inward matter flow starts, with the gas sinking in the potential well of the accretor. More gravitational potential energy is converted into kinetic energy, and more kinetic energy is converted into heat. The heat is finally released as thermal radiation, with the inner regions which are hotter than the outer regions, because of the energy release of gravitational potential energy. The thermal radiation is then detected from the distant observer.

Vertical disc structure

So far we have discussed the horizontal structure of a thin accretion disc. The vertical structure can be calculated very easily by assuming that the gas is in hydrostatic equilibrium along the vertical coordinate z. This means that the vertical component of the gravitational force of the compact object has to be equal to the vertical pressure gradient. Therefore:

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GM}{r^2}\frac{z}{r} \tag{10}$$

with $z \ll r$. Replacing the differentials with finite differences (that is, setting $\Delta P \approx P$, where P is the pressure at $z = 0$, and setting $\Delta z = h$) yields:

$$h = \left(\frac{P}{\rho}\right)^{1/2}\left(\frac{r^3}{GM}\right)^{1/2} \approx \frac{c_s}{\Omega} \tag{11}$$

where c_s is the sound speed in the disc midplane, and where we have used the definition of the sound speed $c_s = \sqrt{P/\rho}$.

Appendix: The Rayleigh Stability Criterion

Consider a fluid of uniform density rotating differentially around its axis of symmetry (like an accretion disc). To find the condition for its stability, we suppose that a fluid ring at a distance r_0 from the axis moving with velocity v_0 is interchanged with a fluid ring at a greater distance r_1 (i.e., $r_1 > r_0$) moving with velocity v_1. The system is stable if the displaced fluid rings tend to return to their initial positions, whereas it is unstable if the displaced rings move further away. Assuming conservation of angular momentum, we conclude that the fluid ring displaced from r_0 to r_1 acquires a velocity $(r_0/r_1)v_0$. The ring previously at r_1 had a centripetal acceleration v_1^2/r_1, which must have been provided by the various forces there such as the part of the pressure gradient left after balancing gravity. The ring brought to r_1 now acquires a centripetal acceleration $r_0^2 v_0^2/r_1^3$ to remain in its new position. If this is less that v_1^2/r_1, then we expect the forces present there to push the ring inward towards its initial position. The condition for stability is then

$$\frac{r_0^2 v_0^2}{r_1^3} < \frac{v_1^2}{r_1} \tag{12}$$

so that:

$$\left(r_0^2 \Omega_0\right)^2 < \left(r_1^2 \Omega_1\right)^2 \tag{13}$$

where $\Omega_n r_n = v_n$ is the angular velocity at radius r_n. This stability condition can be rewritten as:

$$\frac{d\left(r^2 \Omega\right)^2}{dr} > 0 \tag{14}$$