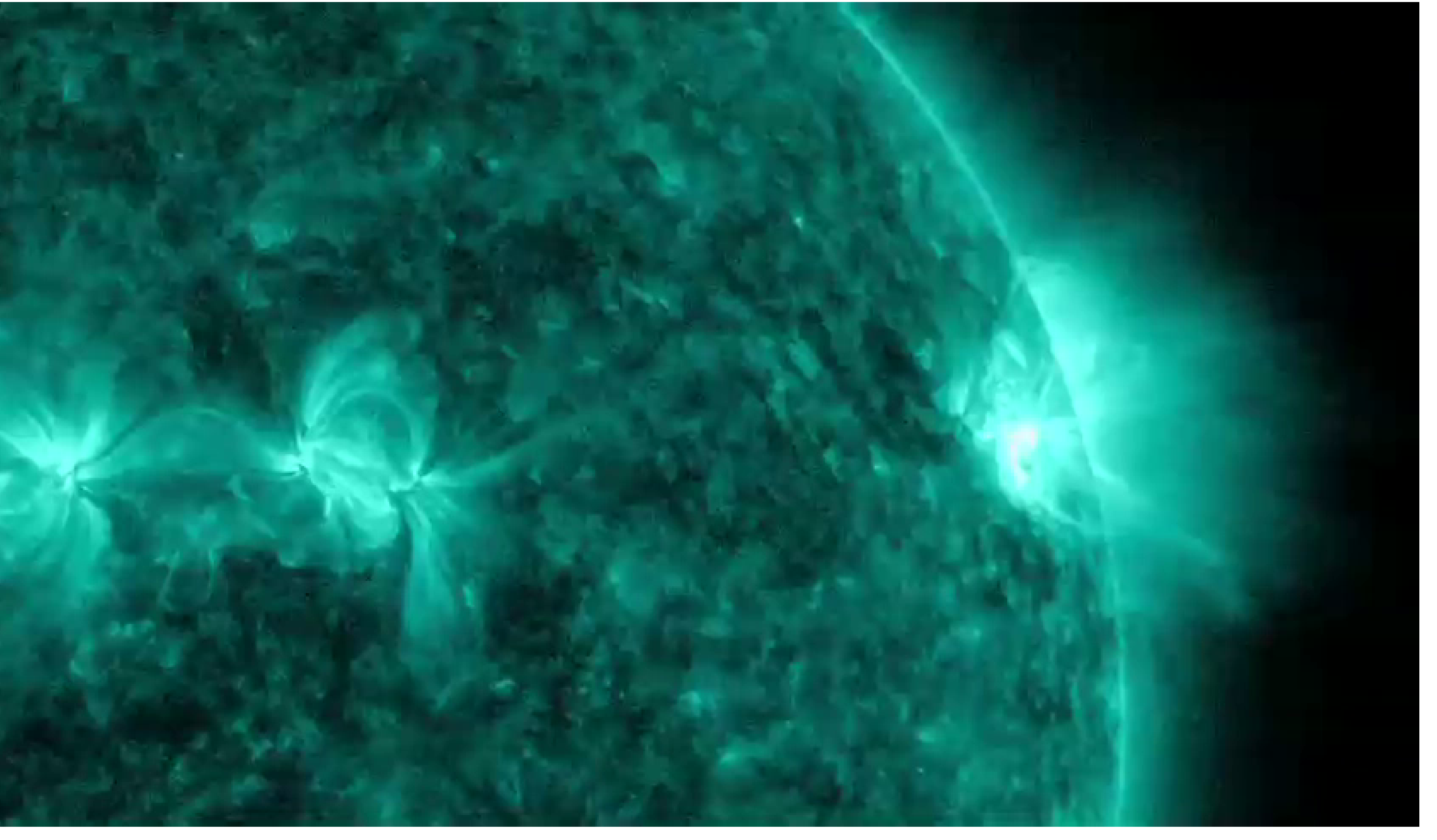
Bremsstrahlung Radiation

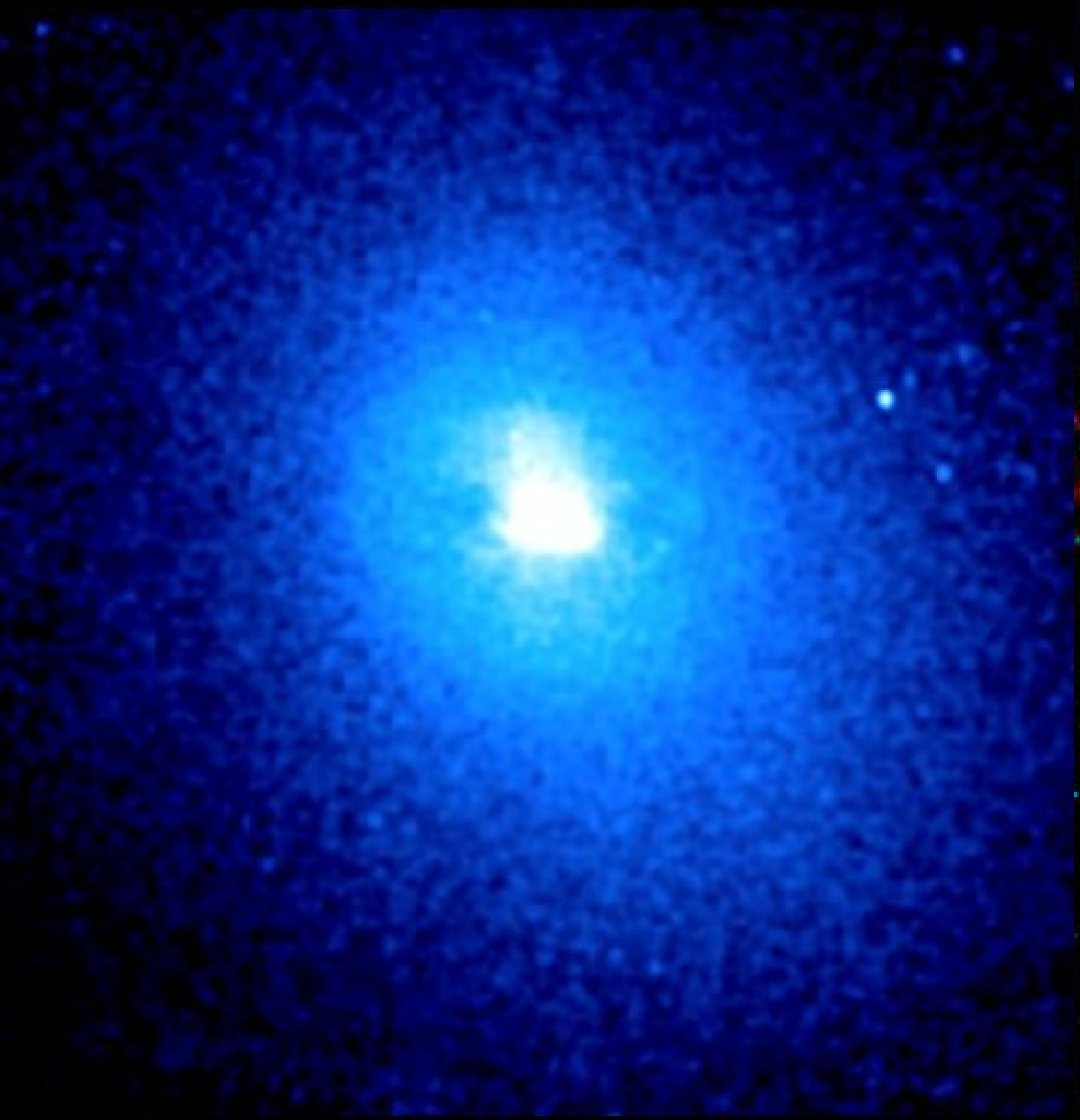


Thermal Radiation Properties (so far)

	Thermal	Blackbody
Optically thick	–	YES
Maxwellian distribution of velocities	YES	YES
Relativistic speeds	–	–
Main Properties	Matter in thermal equilibrium	Matter AND radiation in thermal equilibrium

Abell 2199

Chandra (X-ray)

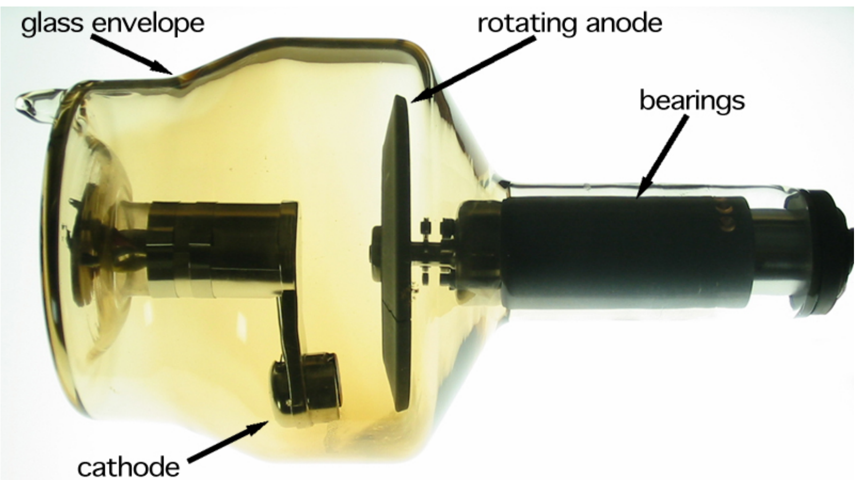


redshift, z = 0.0309

Wise (IR)

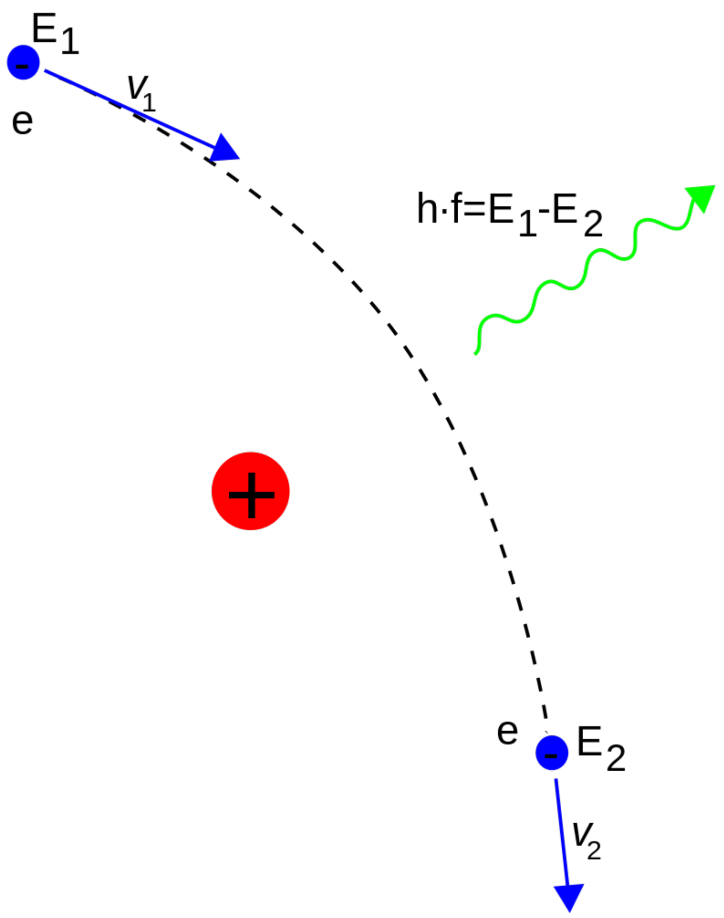
50 thousand light years

An Example in Everyday Life



X-Rays used in medicine (radiographics)
are generated via the Bremsstrahlung process.

In a nutshell:



Bremsstrahlung radiation is emitted when a charged particle is deflected (decelerated) by another charge.

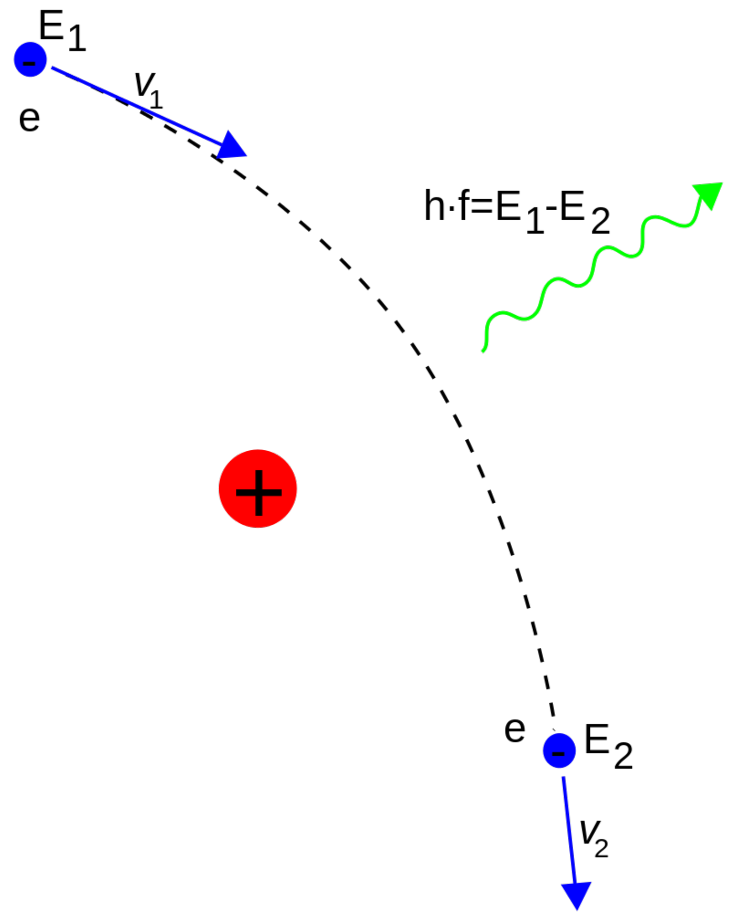
If plasma produces radiation in this way and the radiation can escape the environment without further interaction (i.e., the plasma is optically thin) then you will see Bremsstrahlung radiation.

This type of radiation is seen *very often* in astrophysical phenomena:

- Intracluster medium (X-rays)
- solar flares (X-rays)
- isolated neutron stars (X-rays)
- neutron star binaries (X-rays)
- black hole binaries (X-rays)
- supermassive black holes (X-rays)
- HII regions in the Milky Way (radio)
- Astrophysical Jets (radio)

Bremsstrahlung Radiation

Bremsstrahlung seems a simple process, but in reality is complicated because the energy of the radiation emitted might be comparable to that of the electron producing it.



This means we need a quantum treatment. Particles can also move relativistically, so we might need a relativistic treatment.

However, a classical treatment works well in most cases and the quantum corrections can be introduced as corrections (Gaunt factor)

Finally, the relativistic treatment will be seen later as a special case.

Accelerations: Retarded Potentials

We know from Maxwell equations that the field **E**(r,t) and **B**(r,t) can be expressed in terms of two potentials, $\phi(r,t)$ and **A**(r, t).

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A} \qquad \boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

We also know that the two potentials satisfy the equations:

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho_e}{\varepsilon_0} .$$

And we know that the solution for these two equations is:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|}\, \mathrm{d}^3\boldsymbol{r}' ,$$

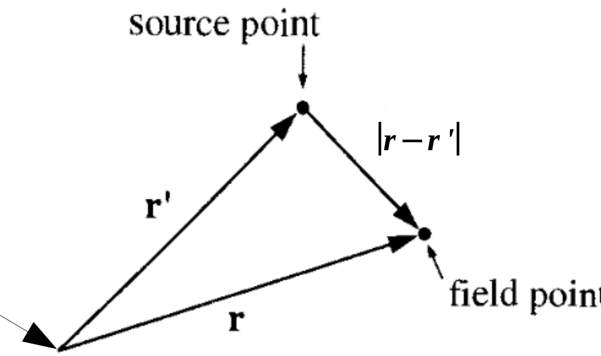
$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_e(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|}\, \mathrm{d}^3\boldsymbol{r}' .$$

Here ***r*** is the point at which the fields are measured. The integration is over the electric current and charge distributions throughout space
The terms |***r-r'***|/*c* take account of the fact that the current and charge distributions should be evaluated at *retarded times*.

Retarded Time

First of all let's clarify what **r** and **r'** are.
In electrodynamics one frequently encounters problems involving two points, typically, a source point, **r'**, where an electric charge is located, and a field point, **r**, at which you are calculating the electric or magnetic field.

This is the origin of our Cartesian reference frame.
We are measuring the field at the distance **r** from the origin of our frame. And we are measuring this field that is generated by the charge at the point **r'**.



The *retarded time* refers to the conditions at the point **r'** that existed at a time *earlier* than *t* by just the time required for light to travel between **r** and **r'**.

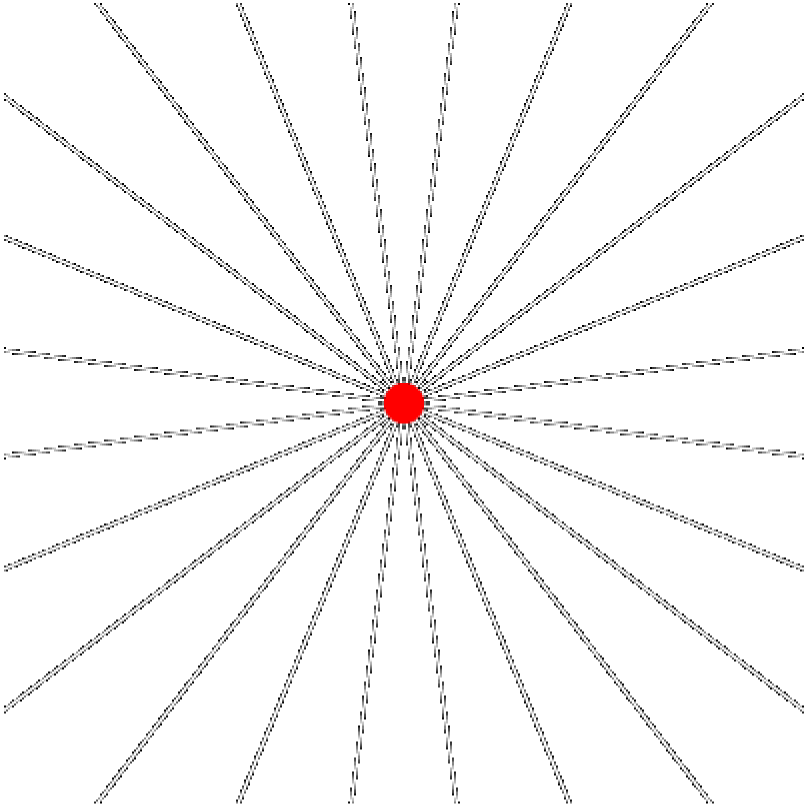
In other words: the field at a certain point in space is not determined by where the charge is NOW (time *t*) but it depends on the state of the charge in the past. How far in the past? Just the time it takes to the fields to propagate from the charge to the point we're measuring.

(If the Sun turns off, you will realize it 8 minutes later).

Accelerations: Retarded Potentials

Electric and magnetic fields move at the speed of light, which is finite.

If you take a charge and move it, the field lines will change. The disturbance will take time to propagate. *It is this "retardation" that makes possible for a charge to radiate!*



See animation!

Velocity and Acceleration Fields

If one calculates the **E** and **B** fields from the retarded potentials one finds the following:

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right],$$

$$\mathbf{B}(\mathbf{r},t) = \left[\mathbf{n}\times\mathbf{E}(\mathbf{r},t)\right].$$

Here **u** is the velocity of the charge, **n** is a unit vector from the charge to the field point.

$$\boldsymbol{\beta} \equiv \frac{\mathbf{u}}{c}, \qquad \kappa \equiv 1 - \mathbf{n}\cdot\boldsymbol{\beta}$$

We have aso used the notation **R = |r-r'|** (so that **n** = **R**/R)

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$$\mathbf{B}(\mathbf{r},t) = [\mathbf{n}\times\mathbf{E}(\mathbf{r},t)].$$

This field falls off as 1/R^2, it is called the *velocity field* and it is a generalization of the Coulomb law for moving particles. For **u**<<c then it becomes precisely Coulomb's law. Note that there is no acceleration in this term, i.e., this field is generated by charges at rest or with constant velocity.

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When **u**~c then the term *k* becomes very important and concentrates the fields in a narrow cone (*beaming effect,* see previous lecture).

Velocity and Acceleration Fields

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$$\boldsymbol{\beta}\equiv\frac{\mathbf{u}}{c}, \qquad \kappa\equiv 1-\mathbf{n}\cdot\boldsymbol{\beta}$$

We have aso used the notation **R = |r-r'**

This is the *acceleration field, i.e.*, it appears when the charges are accelerated.
Note that it falls off as 1/R, not as 1/R^2.
The *acceleration field* is also known as the *radiation field* and it is orthogonal to **n**.

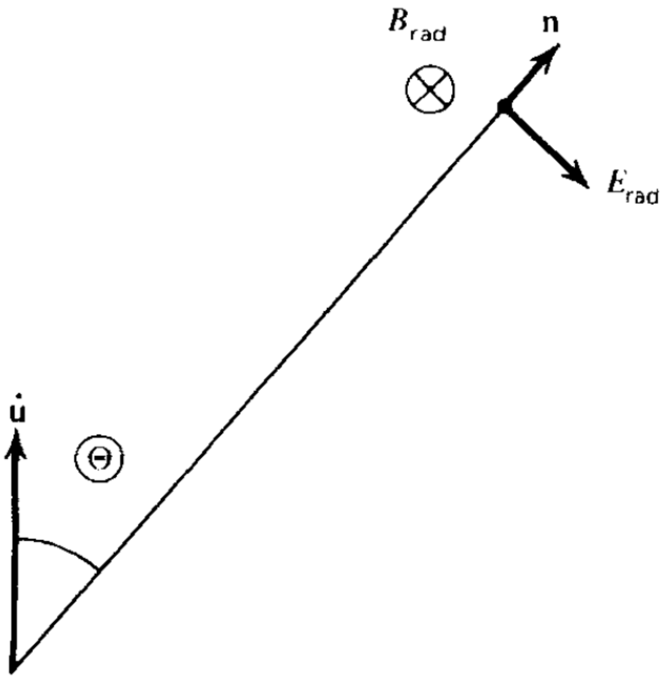
Larmor's Formula

What can we say about the radiation field when the velocity is <<c? (non-relativistic case)

In this case beta<<1 and thus we can simplify the electric and magnetic field expressions and obtain:

$$\mathbf{E}_{\mathrm{rad}} = \left[(q/Rc^2)\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}) \right]$$

$$\mathbf{B}_{\mathrm{rad}}$$



What is the Poynting vector **S**? (remember that the Poyinting vector defines the direction towards which the energy carried by the em fields is directed. Here **S** is parallel to **n;**
S has units of erg/s/cm^2, i.e., energy flux).

Since:

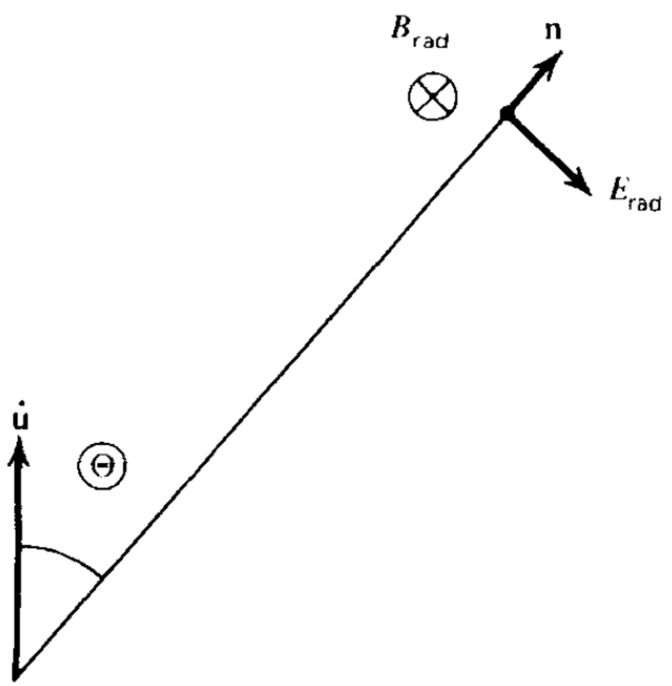
$$|\mathbf{E}_{\mathrm{rad}}| = |\mathbf{B}_{\mathrm{rad}}| = \frac{q\dot{u}}{Rc^2}\sin\Theta$$

The Poynting vector has magnitude:

$$S = \frac{c}{4\pi}E_{\mathrm{rad}}^2 = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta.$$

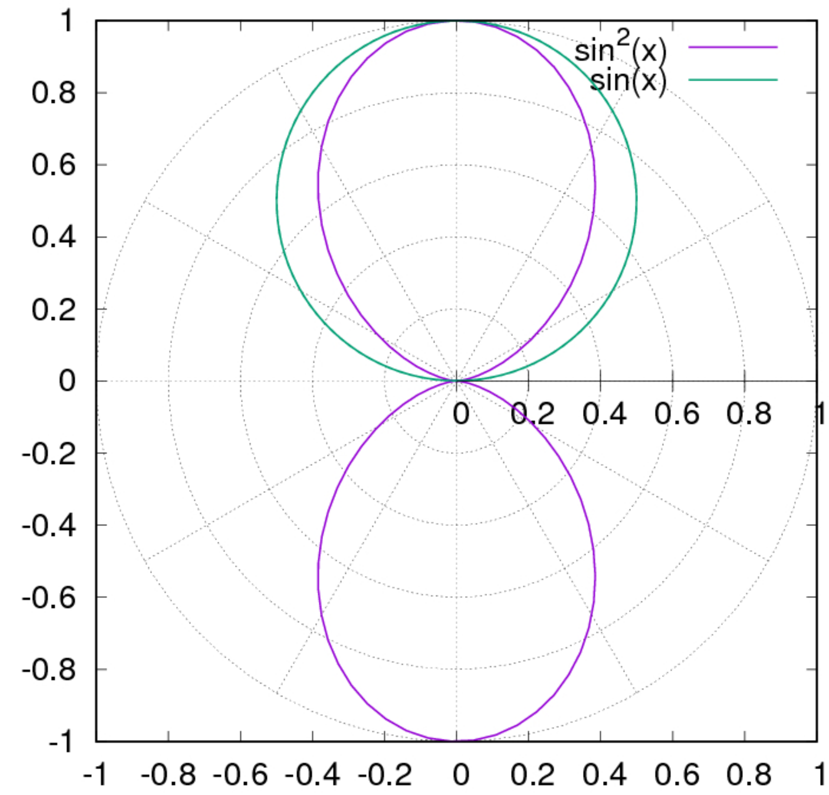
Larmor’s Formula

$$S = \frac{c}{4\pi} E_{\mathrm{rad}}^2 = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^4} \sin^2 \Theta.$$



Note the angle theta!

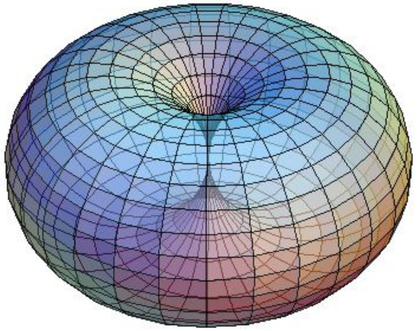
The energy of the em. field is not isotropic but there is a sin^2 !!



Larmor's Formula

Now let's calculate the power in a unit solid angle about **n**. To do this we multiply the Poynting vector (units: erg/s/cm^2) by an area dA (cm^2) to get a power (erg/s). How do we choose dA? We know that the solid angle dOmega = dA/R^2. Therefore:

$$\frac{dW}{dt\, d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2 \Theta$$

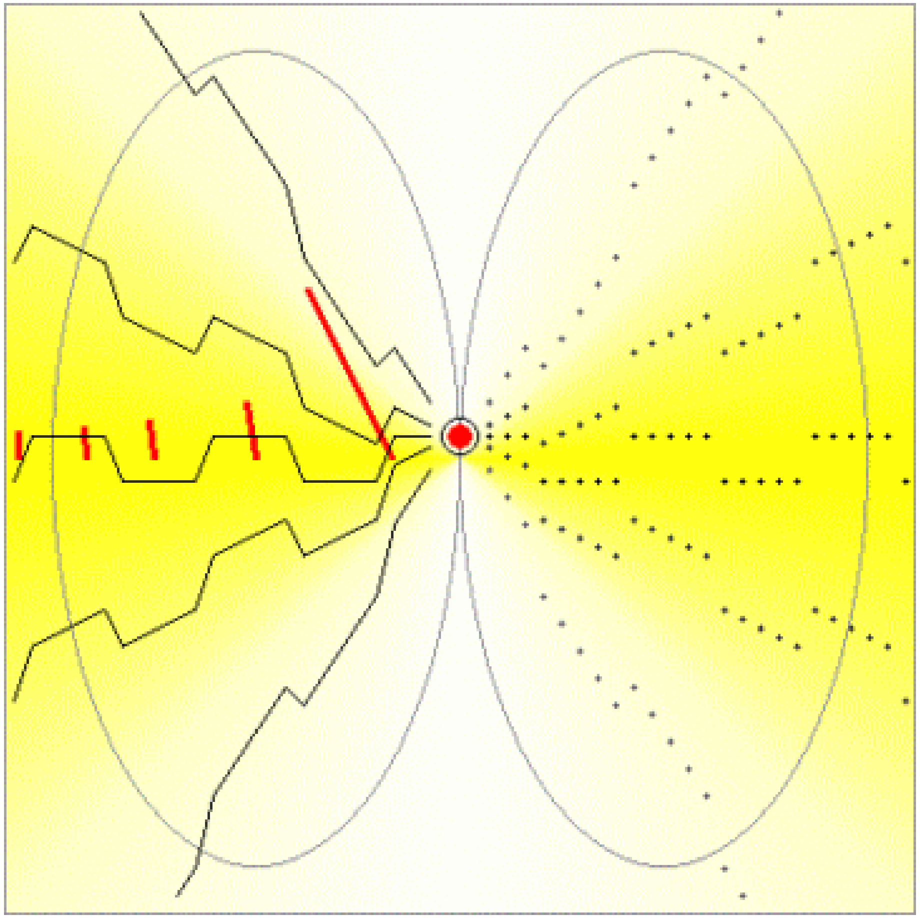


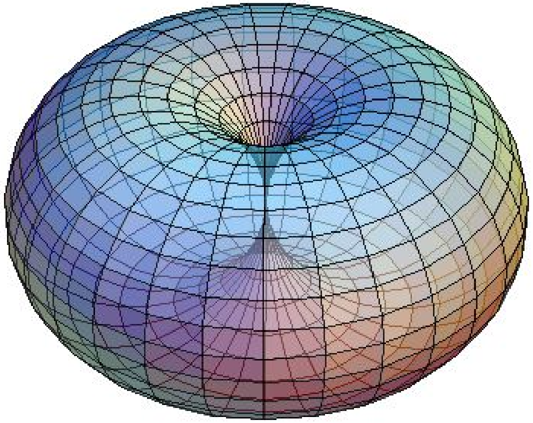
And now we integrate the above expression over the whole solid angle Omega=4*pi and we obtain the total power emitted by an accelerated charge in the non-relativistic approximation:

$$P = \frac{2 q^2 \dot{u}^2}{3 c^3}$$

Larmor's Formula

IMPORTANT: The power emitted is proportional to <u>the square of the charge</u> and <u>the square of the acceleration</u>.



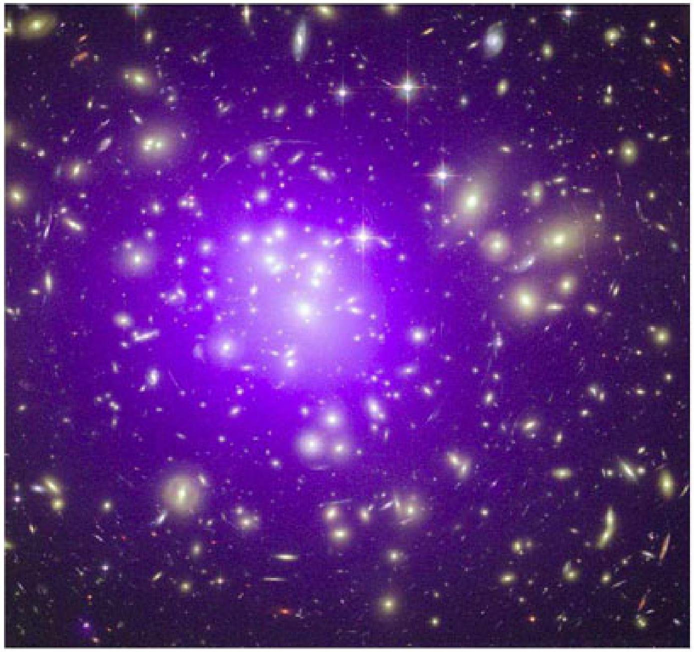


The animation represents a charged particle being switched up and down in a very strong electric field, such that the shape is traced out in time and aligns to an approximate square wave. The ovals' reference lines are drawn to the left and right of the charge and correspond to a cross-section through the doughnut toroid, as illustrated in the previous diagram. Based on the criteria of the Larmor formula, when a charge is subject to acceleration, i.e. during the transition positions, it radiates power also subject to the angle θ with respect to the axis of charge motion. As such, the energy density is reflected by the depth of the yellow shading, symmetrical about the axis of motion. However, the intention of the left-right sides of the animation is to be somewhat illustrative of wave-particle duality in that the left reflects the electric field lines, while the right reflects the streams of photons being emitted by the charge. The field lines or photon streams are shown at different angles, e.g. 0, 30, and 60 degrees, from the maximum, which is always perpendicular to the axis. Finally, the oscillating red lines on the left reflect the total electric field E=E_rad + E_vel as a function of distance. So what you see is the effects of E_vel reducing by 1/R^2, while E_rad only reduces by 1/R and so quickly becomes the dominant field as the radius from the charge increases.

Ensemble of Particles

So far so good, but what about an ensemble of particles? After all if we want to calculate the properties of Bremsstrahlung radiation we need to consider a lot of particles...

There is a complication here, because the expressions for the radiation fields refer to conditions at retarded times, and these retarded times will differ for each particle and we have an *enormous* amount of particles...



Abell 1689 (z=0.18, i.e., about 2 billion light years away)

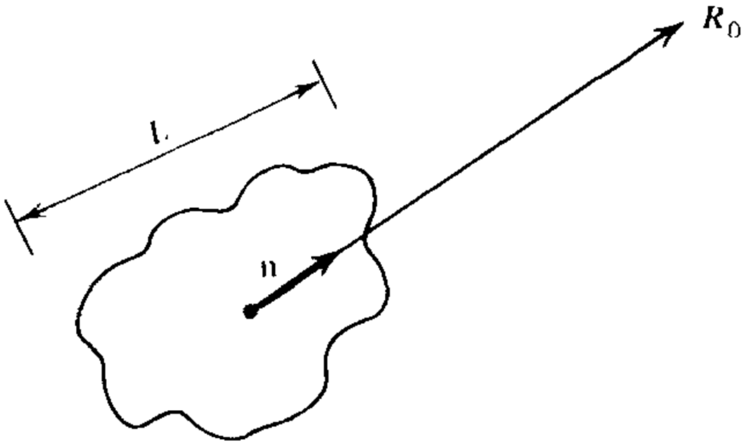
Solution: Let the typical size of the system be L, and let the typical time scale for changes within the system be T. If T is much longer than the time it takes light to travel a distance L, T>>L/c, then the differences in retarded time across the source are negligible.

Does this happen for example in an intra-cluster plasma?

Ensemble of Particles

Now our radiation field is:

$$\mathbf{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i}$$



Of course we have *no idea* what are the single velocities of each particle, neither we know how many particles there are!

Solution: call L the size of our cluster.
Call R0 the distance from some point in the system to the field point (i.e., where we are since we are measuring this field).
But now you see that the difference between each Ri tends to zero as R0 → infinity (since we are very far away from the cluster!). So we can write:

$$\mathbf{E}_{\text{rad}} = \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})}{c^2 R_0}$$

where $\mathbf{d} = \sum_i q_i \mathbf{r}_i$. is the dipole moment of the charges.

Dipole Approximation

Following the same procedure as for the single particle case, we can find the total power emitted by an ensemble of particles (in the **non-relativistic** limit) in the so-called *dipole approximation:*

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta$$

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}.$$

Using Fourier transform we can easily find that:

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 \hat{d}(\omega) e^{-i\omega t} d\omega,$$

$$\hat{E}(\omega) = -\frac{1}{c^2 R_0} \omega^2 \hat{d}(\omega) \sin \Theta.$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\hat{d}(\omega)|^2.$$

Now we have the key to understand Bremsstrahlung radiation...

Dipole Approximation

Following the same
emitted by an ense
in the so-called *dip*

Remember how the Fourier Transform of a derivative works:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$$

$$f'(t) = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\, F(\omega)\, e^{i\omega t} d\omega$$

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Dipole Approximation

This term instead comes from:

$$\mathbf{E}_{\mathrm{rad}} = \frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{d}})}{c^2 R_0}$$

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0}$$

case, we can find the total power
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This comes from Parseval's Theorem.

$$\int_{-\infty}^{\infty} E^2(t)\,dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega$$

Total energy per unit area in a pulse:

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\,dt.$$

$^2\Theta$

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 \hat{d}(\omega) e^{-i\omega t} d\omega,$$

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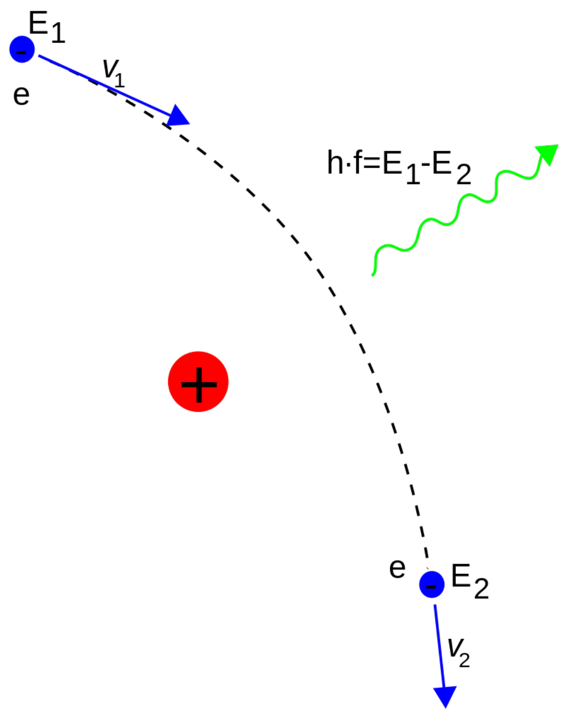
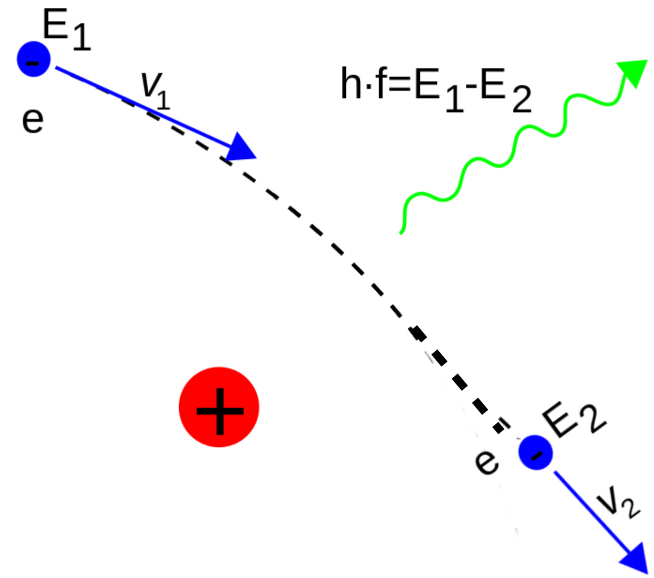
Small-Angle Scattering

To derive the properties of Bremsstrahlung radiation we will use an approximation called *small-angle scattering.* This is an approximation in which the electron deflected by an ion deviates only by a small angle (typically <10 degrees).

Small angle scattering NOT VALID

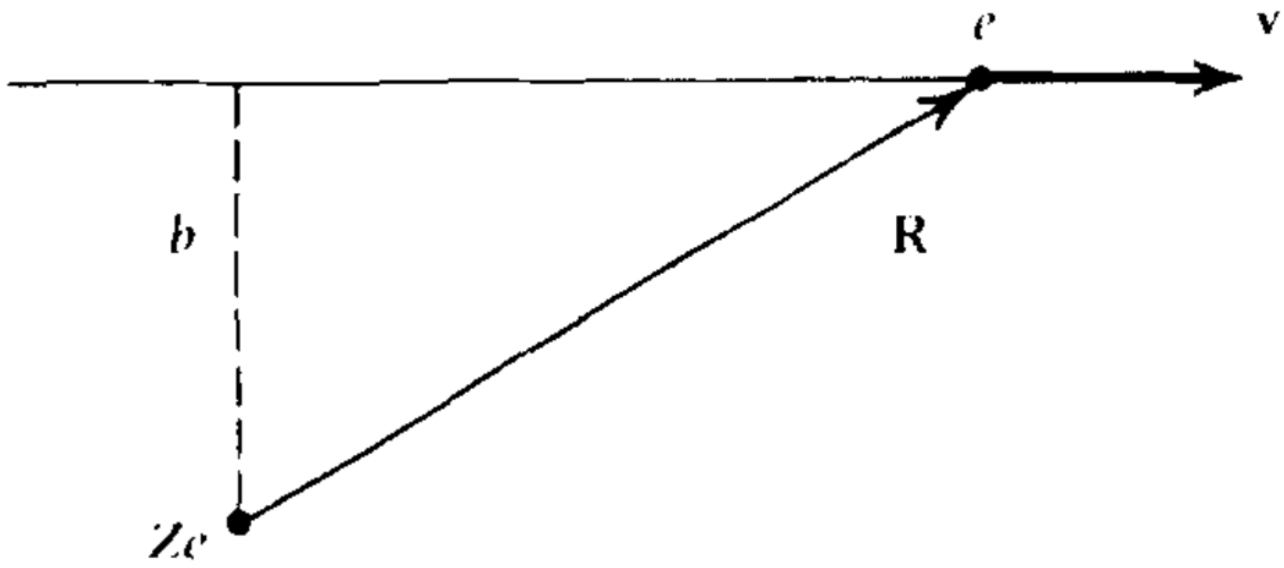


Small angle scattering



This approximation is *not necessary* but it simplifies the calculations and gives the right equations.

Small Angle Scattering



b is the impact parameter (i.e., the perpendicular distance between the path of the electron and the ion of charge Ze). R is the actual distance between the electron and the ion.
v is the speed of the electron.
The dipole moment **d**=-e**R**. Therefore its second derivative is:

$$\ddot{\mathbf{d}} = -e\dot{\mathbf{v}}$$

Small Angle Scattering

Now let's take the Fourier transform of the second derivative of the dipole moment. This is:

$$-\omega^2\hat{\mathbf{d}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty}\dot{\mathbf{v}}e^{i\omega t}dt$$

(Remember that $e^{ix} = \cos x + i\sin x$)

From the dipole approximation we know that the total energy emitted per unit frequency is:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2.$$

So we need to solve the Fourier transform above and we will know what is the energy emitted per unit frequency.

The electron interacts with the ion only for a small amount of time of the order of:

$$\tau = \frac{b}{v}$$ (collision time)

Therefore we can write:

$$\hat{\mathbf{d}}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2}\Delta\mathbf{v}, & \omega\tau \ll 1 \\ 0, & \omega\tau \gg 1 \end{cases}$$

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$\omega\tau \ll 1$ → (the exponential is unity)

$\omega\tau \gg 1$ → (the exponential is zero)

Small Angle Scattering

$$\hat{\mathbf{d}}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2}\Delta\mathbf{v}, & \omega\tau \ll 1 \\ 0, & \omega\tau \gg 1 \end{cases} \longrightarrow \frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2 \longrightarrow \frac{dW}{d\omega} = \begin{cases} \frac{2e^2}{3\pi c^3}|\Delta\mathbf{v}|^2, & \omega\tau \ll 1 \\ 0, & \omega\tau \gg 1. \end{cases}$$

So the energy emitted per unit frequency depends on the change of the electron velocity during the collision time.

Now, we have an energy per unit frequency. But what we really want is the radiated power per unit volume per unit frequency. Remember that for an isotropic emitter:

$$j_\nu = \frac{1}{4\pi} P_\nu;$$

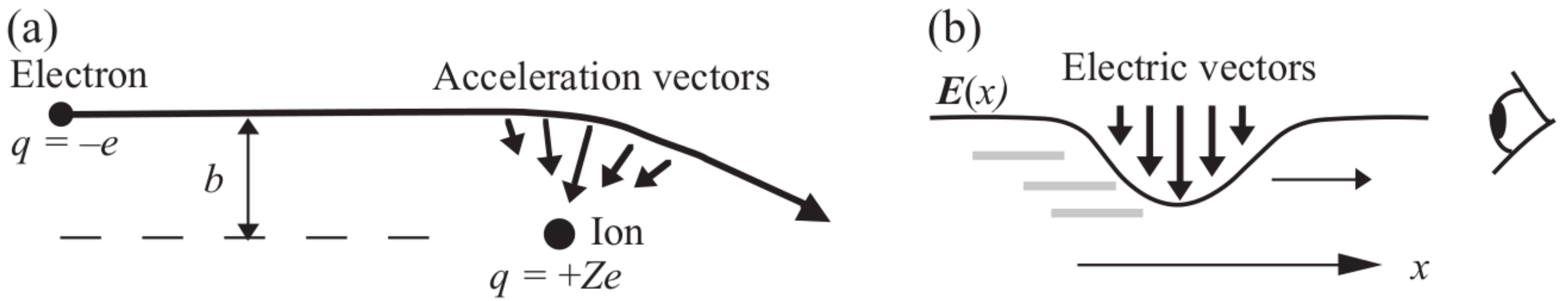
where P_nu was the power (i.e., energy per unit time) per unit volume per unit frequency.

So we want to find here the energy per unit time per unit volume per unit frequency as well.

$$\frac{dW}{d\omega} \rightarrow \frac{dW}{d\omega\, dV\, dt}$$

How do we do this last step?

First we calculate how much has the speed changed (Δv) , so we know $\frac{dW}{d\omega}$



The acceleration (change in velocity) is given by the Coulomb force:

$$\dot{v}=\frac{F}{m}=\frac{Ze^2}{mb^2}$$

where I have used the fact that the interaction occurs at R~b only.

Then we multiply this acceleration by the collision time and we find Δv

$$\Delta v \approx \dot{v}\,\tau = \dot{v}\,\frac{b}{v} = \frac{Ze^2}{mbv}$$

Therefore $\frac{dW}{d\omega}$ depends on the impact parameter b, i.e., it is

$$\frac{dW}{d\omega} \rightarrow \frac{dW(b)}{d\omega} \propto \frac{Z^2 e^6}{m^2 v^2 b^2}$$

Spectrum of an ensemble of particles with a single velocity **v**

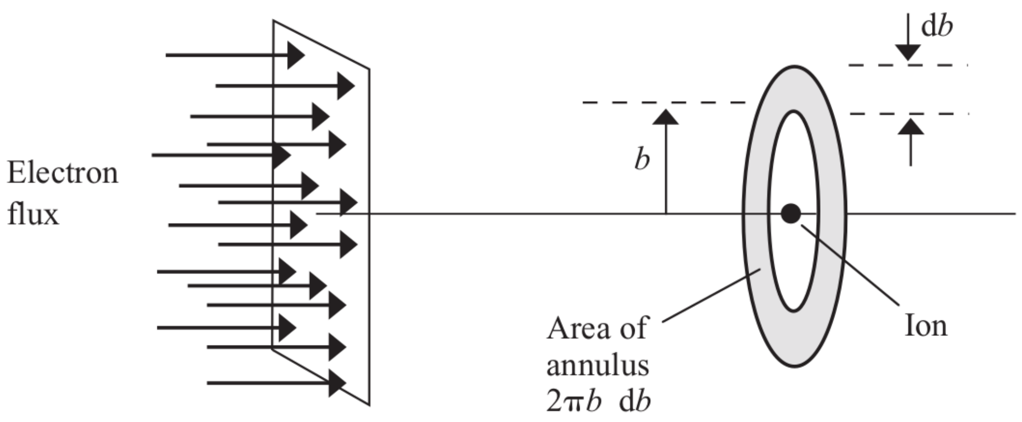
To find the spectrum $\frac{dW}{d\omega\, dV\, dt}$ of an ensemble of particles <u>with a single velocity **v**</u> we need to first integrate over the impact parameter b, then divide by the unit volume and time.

Now, say that the plasma has a certain electron density *ne* and ion density *ni,* and that **all** the electrons have the same speed **v**. The area around each ion that is important for the interaction is:

$$dA = 2\pi\, b\, db$$

$$\frac{dW}{d\omega\, dV\, dt} = n_e n_i 2\pi v \int_{b_{min}}^{\infty} \frac{dW(b)}{d\omega}\, b\, db$$

(units of 1/volume)



Spectrum of an ensemble of particles with a single velocity **v**

Now the treatment on how to choose the boundaries of the integration becomes quite lengthy and complicated. We are interested only in a few features that will determine how the final spectrum will look like.

The final spectrum of plasma with electron having a single velocity **v** will look like the following:

$$\frac{dW}{d\omega\, dV\, dt} = \frac{16\pi e^6}{3\sqrt{3}\; c^3 m^2 v} n_e n_i Z^2 g_{ff}(v,\omega).$$

$$\left(P_\nu = \frac{j_\nu}{4\pi} = \frac{dW}{d\nu\, dV\, dt} \right)$$

The Gaunt factor contains quantum corrections which we have not taken properly into account here, but can be approximated as:

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{\max}}{b_{\min}} \right)$$

Thermal Bremsstrahlung

How do we go from the spectrum of an ensemble of ions and electrons (with the latter all having a single velocity **v)** to the spectrum of an ensemble of ions and electrons with a distribution of velocities?

First we need to know *which distribution* of velocities.

Let's take the most common case (almost always the case in astrophysics) which is that of a thermal plasma, i.e., electrons and ions with velocities distributed according to the Maxwell-Boltzmann distribution (see also Lecture 3!).

$$F(v)\,dv = 4\pi v^2 \left(\frac{m}{2\pi k T}\right)^{3/2} e^{-mv^2/2kT}\,dv$$

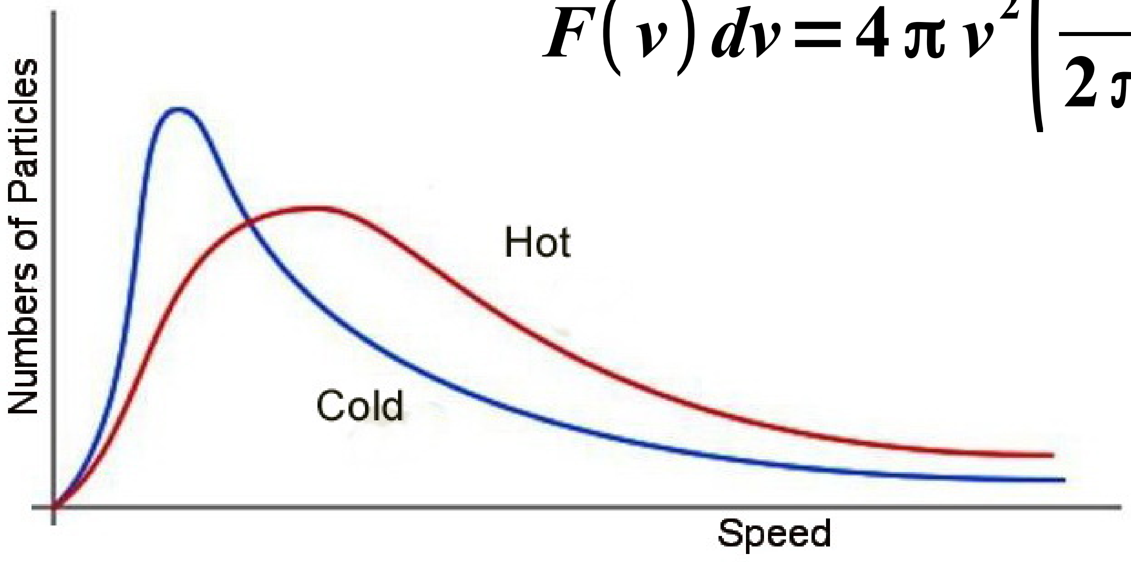
We then need to integrate $\frac{dW}{d\omega\,dV\,dt} = \frac{16\pi e^6}{3\sqrt{3}\,c^3 m^2 v} n_e n_i Z^2 g_{ff}(v,\omega).$ over this distribution

From Lecture3: Matter in Thermal Equilibrium

Suppose to have a plasma in thermal equilibrium (*thermal plasma*). What does this mean in terms of micro-physical properties of the matter?

Probability distribution function of (non-relativistic) velocities is the Maxwell-Boltzmann distribution:

$$F(v)\,dv = 4\pi v^2 \left(\frac{m}{2\pi k T}\right)^{3/2} e^{-mv^2/2kT}\,dv$$



Spectrum:Thermal Bremsstrahlung

$$\frac{dW(T,\omega)}{dV\,dt\,d\omega} = \frac{\displaystyle\int_{v_{\min}}^{\infty} \frac{dW(v,\omega)}{d\omega\,dV\,dt}\, v^2 \exp(-mv^2/2kT)\,dv}{\displaystyle\int_0^{\infty} v^2 \exp(-mv^2/2kT)\,dv}$$

Why there is a minimum velocity in the integral? Shouldn’t we use zero as the minimum?

Spectrum:Thermal Bremsstrahlung

$$\frac{dW(T,\omega)}{dV\,dt\,d\omega} = \frac{\int_{v_{\min}}^{\infty} \frac{dW(v,\omega)}{d\omega\,dV\,dt} v^2 \exp(-mv^2/2kT)\,dv}{\int_0^{\infty} v^2 \exp(-mv^2/2kT)\,dv}$$

Why there is a minimum velocity in the integral? Shouldn't we use zero as the minimum?

The photons need to be created during the deceleration of the electron. So the initial kinetic energy of the electron *must be* larger than the photon energy.
This creates a cutoff in the spectrum and this is due to the *discreteness of photons*, i.e., they are discrete and not continuum entities.

$$v_{min} = (2h\nu/m)^{1/2}$$

Spectrum:Thermal Bremsstrahlung

Performing the integration one gets:

$$\varepsilon_\nu^{ff} = \frac{dW}{d\nu\, dV\, dt} = 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}$$

BE CAREFUL do NOT make a confusion between ε_ν^{ff} and ϵ_ν defined as the *emissivity* at page 9.
Also, R&L uses the same symbol ϵ_ν to define the probability of absorption at page 37.

Furthermore the difference between ε_ν^{ff} and j_ν is the following:

ε_ν^{ff} → (energy/frequency/volume/time).

j_ν → (energy/frequency/volume/time/*solid angle*).

Also, the symbol ε_ν^{ff} is exactly the same as P_ν in $j_\nu = \frac{1}{4\pi} P_\nu$;

The reason why R&L uses different symbols here is correct: ε_ν^{ff} will refer from now on only to Bremsstrahlung. The symbol P_ν is a general one and it equal to ε_ν^{ff} only for Bremsstrahlung

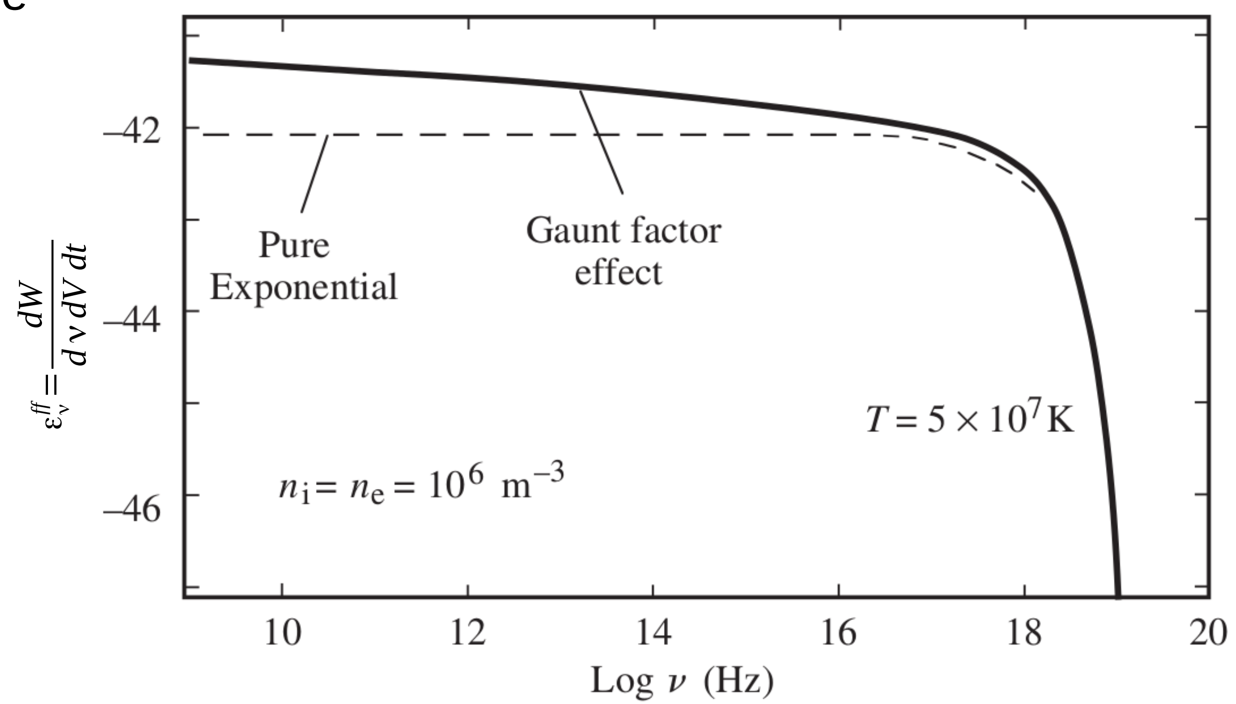
Spectrum:Thermal Bremsstrahlung

Performing the integration one gets:

$$\varepsilon_\nu^{ff} = \frac{dW}{d\nu\, dV\, dt} = 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}$$

What do we see here?
The emission coefficient seem to depend on the temperature (be careful because T is also in the exponential), on the density of ions and electrons and on the ion charge.
The frequency dependency is only in the exponential. The average Gaunt factor can be considered very close to unity since this is its order of magnitude.

Also, the spectrum will be basically flat, except when exp(-h*nu/kT) becomes dominant.
This happens when the thermal energy of electrons is basically insufficient to generate high energy photons.



Thermal Bremsstrahlung: Absorption

What happens at low frequencies?

If we have thermal emission then we can *always* use Kirchhoff’s law.

Thermal Bremsstrahlung: Absorption

What happens at low frequencies?

If we have thermal emission then we can *always* use Kirchhoff's law.

$$j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T)$$

$$\varepsilon_\nu^{ff} = \frac{dW}{d\nu\, dV\, dt} = 4\pi\, j_\nu^{ff} = 6.8\times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}$$

$$\alpha_\nu^{ff} = 3.7\times 10^{8} T^{-1/2} Z^2 n_e n_i \nu^{-3} (1 - e^{-h\nu/kT}) \bar{g}_{ff}$$

We see that when $h\nu \ll kT$ we are in the Rayleigh-Jeans regime:

$$\alpha_\nu^{ff} = 0.018\, T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff}.$$

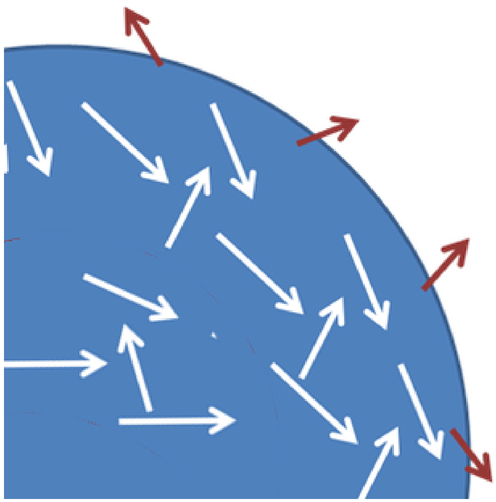
This is telling us that the spectrum of Bremsstrahlung is *self-absorbed* at low frequencies. Why?

Thermal Bremsstrahlung

Remember that the optical depth is defined as: $d\tau_\nu = \alpha_\nu ds$

Therefore since $\alpha_\nu^{ff} = 0.018\,T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff}$. we have that $\tau_\nu \propto \nu^{-2}$ as well.

The smaller the frequencies, the larger the optical depth. This means that radiation is absorbed more and more before leaving the system. But this is precisely what a blackbody is! So at *low frequencies* we expect a blackbody like spectrum.



So what is the *specific brightness* of Bremsstrahlung radiation?

At low frequency we expect it to look like blackbody.
At intermediate frequencies it has to be flat
At high frequencies there must be an exponential cutoff

REMEMBER FROM LECTURE 2:

Equation of transfer

- This yields the format solution of the EOT:

$$I_\nu(\tau_\nu) = I_{\nu,0}\, e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

- When S_ν constant:
  $I_\nu(\tau_\nu) = I_{\nu,0}\exp(-\tau_\nu) + S_\nu(1-\exp(-\tau_\nu))$
- $\tau >> 1$: $I_\nu \rightarrow S_\nu$
- $\tau << 1$: $I_\nu \rightarrow I_{\nu,0} + S_\nu \tau_\nu$

REMEMBER FROM LECTURE 2:

A special case

- When S_ν is constant throughout the source, this can be rewritten as:

$$I_\nu(\tau_\nu) = I_{\nu,0}\, e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right)$$

Question: What is the intensity of this source for small and large optical depth when it has size R?

REMEMBER FROM LECTURE 2:

Answer

- If $I_{\nu,0} = 0$ then $I_\nu(\tau_\nu) = \frac{j_\nu}{\alpha_\nu}\left(1 - e^{-\tau_\nu}\right)$
- A little trick. First, we multiply by the source size s=R:

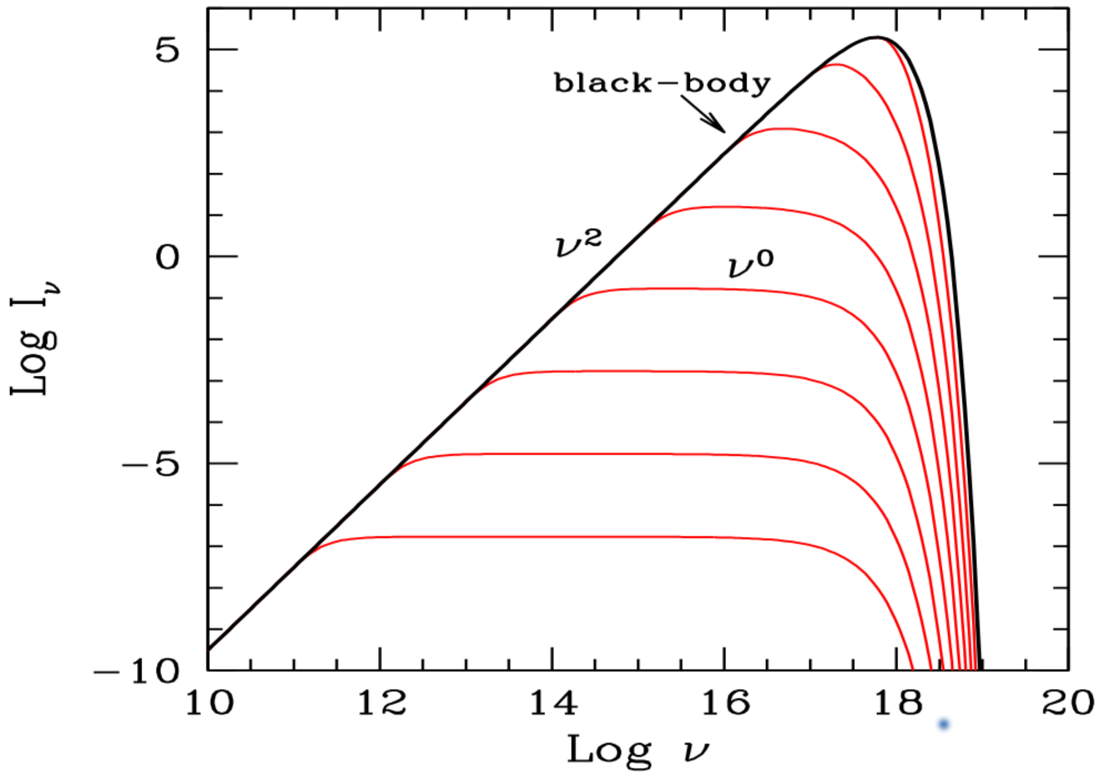
$$I_\nu(\tau_\nu) = \frac{j_\nu R}{\alpha_\nu R}\left(1 - e^{-\tau_\nu}\right) = j_\nu R\left(\frac{1 - e^{-\tau_\nu}}{\tau_\nu}\right)$$

- Optically thin (τ << 1): 1 – exp(-τ) = 1 – 1 + τ = τ

→ $I_\nu(\tau_\nu) = j_\nu R$

- Optically thick (τ >> 1): $I_\nu(\tau_\nu) = \frac{j_\nu R}{\tau_\nu}$

Thermal Bremsstrahlung



Now we can understand the spectrum of Bremsstrahlung!

At large optical depths: Blackbody

$$I_\nu(\tau_\nu) = \frac{j_\nu R}{\tau_\nu} = \frac{j_\nu}{\alpha_\nu} = S_\nu = B_\nu$$

At small optical depths:

$$I_\nu(\tau_\nu) = j_\nu R = \frac{\varepsilon_\nu^{ff} R}{4\pi}$$

$$\varepsilon_\nu^{ff} = \frac{dW}{d\nu\, dV\, dt} = 4\pi\, j_\nu^{ff} = 6.8\times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}$$

If the region of size R has large optical depth at any frequency then Bremsstrahlung becomes Blackbody spectrum (solid line). Otherwise is will show the typical flat spectrum in the intermediate frequencies, blackbody spectrum at low frequencies at cutoff at high frequencies

Summary of Radiation Properties

	Thermal	**Blackbody**	**Bremsstrahlung**	**Synchrotron**	**Inverse Compton**
Optically thick	–	YES	NO		
Maxwellian distribution of velocities	YES	YES	–		
Relativistic speeds	–	–	–		
Main Properties	Matter in thermal equilibrium	Matter AND radiation in thermal equilibrium	Radiation emitted by accelerating particles		

Rules of thumb:
1. Blackbody is always thermal, but thermal radiation is not always blackbody (e.g., thermal Bremsstrahlung)
2. Bremsstrahlung can be thermal or non-thermal.
3. Bremsstrahlung becomes blackbody when optical depth >>1.

Cooling Time

Since we know how much does a thin plasma radiate we can calculate the energy losses and the so-called cooling time:

$$\frac{\text{energy density of the plasma}}{\text{rate of energy loss}} = \frac{\frac{3}{2}(n_e + n_i)kT}{\varepsilon^{ff}} = \frac{3nkT}{\varepsilon^{ff}}$$

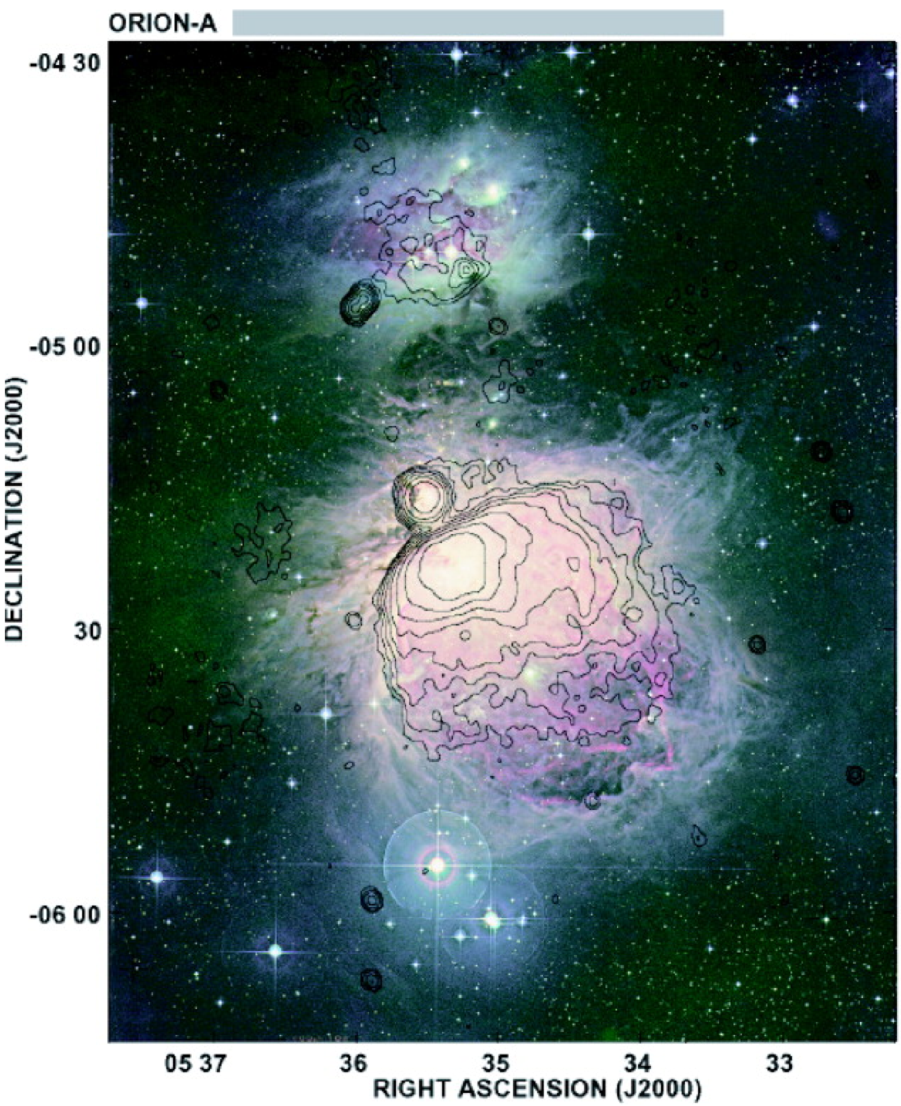
Here we have integrated the emission coefficient ε_ν^{ff} over all frequencies:

$$\varepsilon^{ff} \equiv \frac{dW}{dt\,dV} = 1.4 \times 10^{-27} T^{1/2} n_e n_i Z^2 \bar{g}_B. \longrightarrow L = \varepsilon^{ff} V$$

This is useful to calculate the cooling time:

$$\tau_{cool} = 6 \times 10^3 T^{1/2} n_e^{-1} \bar{g}_{ff}\, yr$$

Cooling Time: HII regions



The Orion nebula is an HII region.
Here you see the radio continuum overlaid to the optical image. The radio continuum is Bremsstrahlung emission.

What is the cooling time of the nebula?

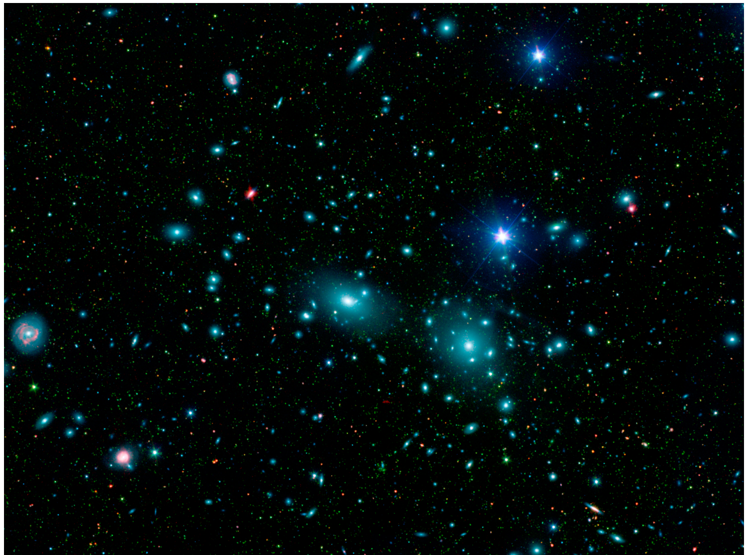
$$\tau_{cool} = 6 \times 10^{3} T^{1/2} n_e^{-1} \bar{g}_{ff}\, yr$$

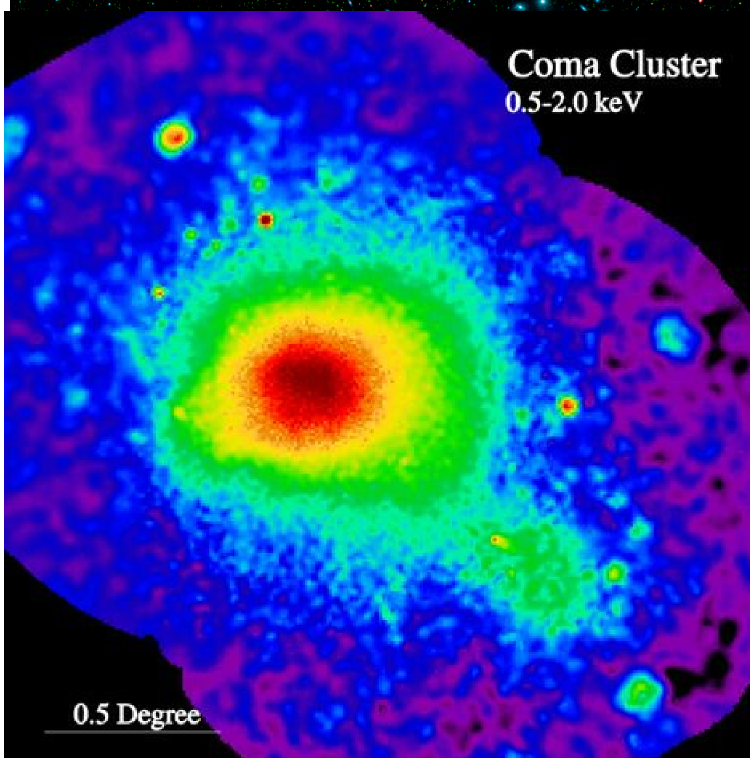
Here ne ~ 100-1000 cm^-3
T ~ 10,000 K

The cooling time is of the order of a few thousands years.

But the nebula has an age of 3 Myr.
So what does this mean?

Cooling Time: Intracluster medium





Here the typical temperatures are 10^7 K (indeed we see most radiation in X-rays, whereas in the Orion nebula it was mostly at radio waves).
The typical densities are also very low:

ne ~ 0.001 cm^-3

$$\tau_{cool} = 6 \times 10^3 T^{1/2} n_e^{-1} \bar{g}_{ff}\, yr \approx 10\, Gyr$$

Intracluster gas takes a very long time to cool down!