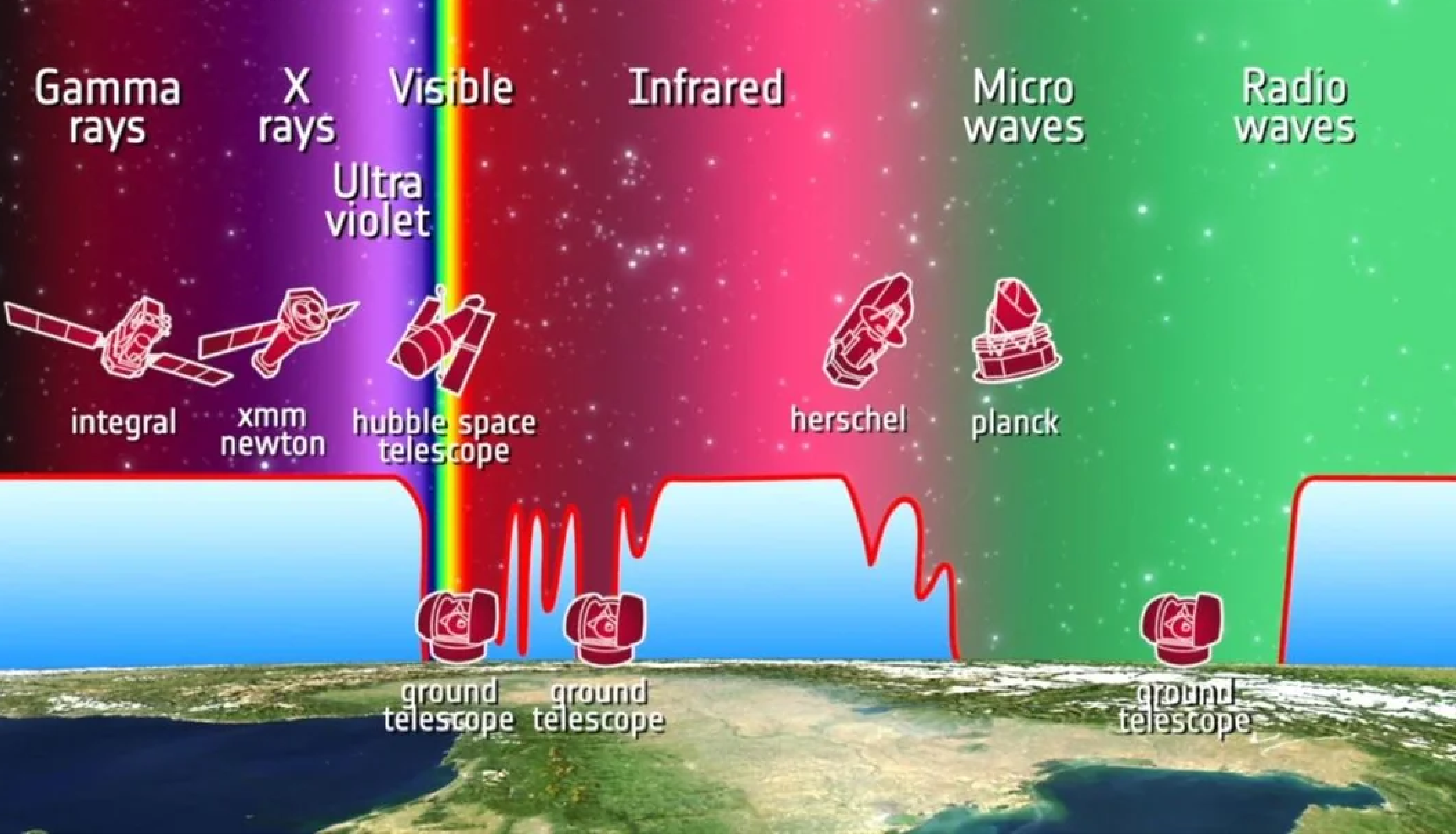
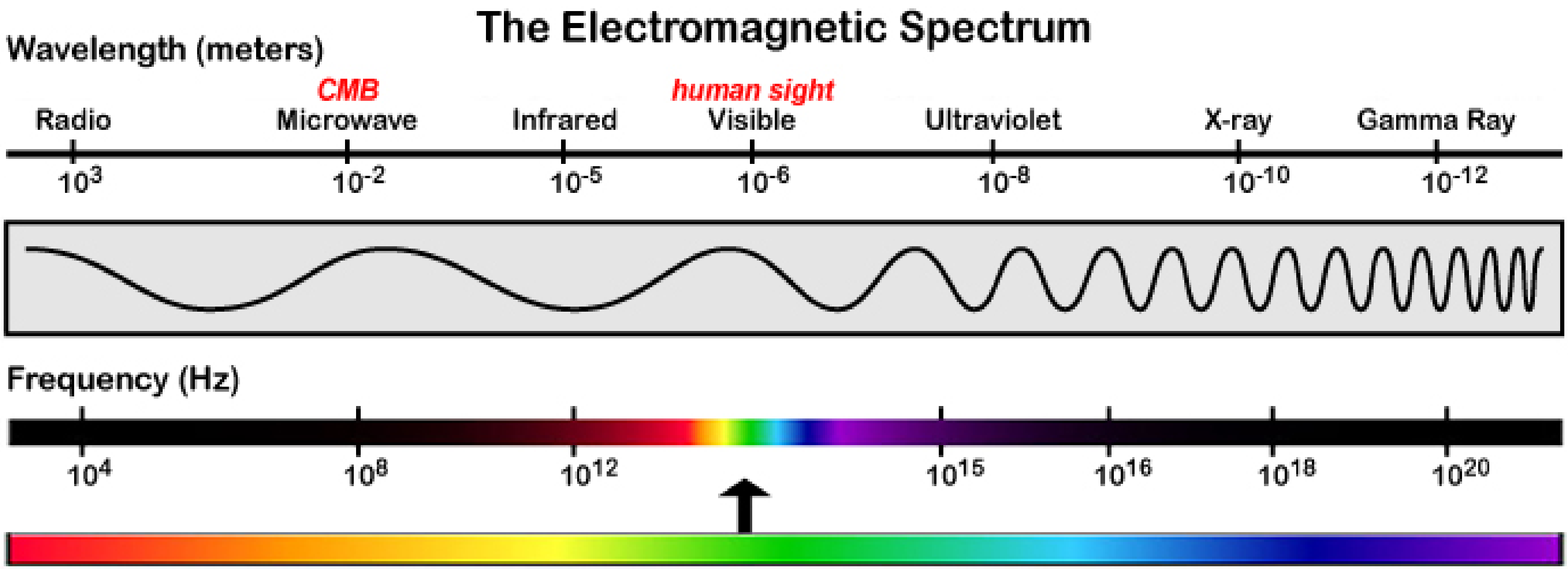


Radiative Transport (Lecture 1)

Introduction to the Physics of the Cosmos

2025



$$\lambda = \frac{c}{\nu}$$

$$E = h\nu = \hbar\omega$$

$$1\,eV \rightarrow 11{,}000\,K \left(E = k_B T \right)$$

Radio $> 1\,\mathrm{mm}$

IR $700\mathrm{nm} - 1\mathrm{mm}$

Optical $400 - 700\mathrm{nm}$

UV $100 - 400\mathrm{nm}$

X-Rays $0.1 - 100\ \mathrm{keV}$

Gamma $> 1\ \mathrm{MeV}$

What is a "ray of light"?

Say a photon has wavelength *lambda* and momentum *p*. From Heisenberg principle we know that the position (encoded in lambda) and momentum of the photon cannot be known with arbitrary precision.

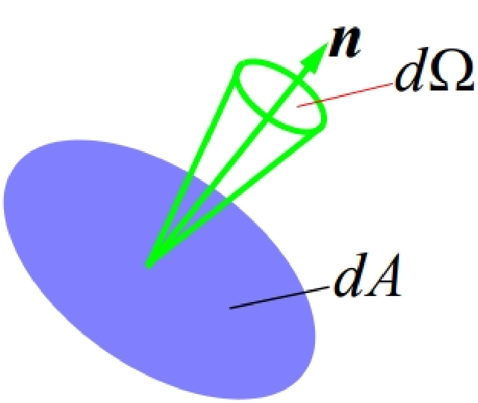
Therefore if we know the wavelength *lambda* with a certain precision, the direction of the photon (encoded in its momentum *p*) cannot be known with arbitrary precision, so the direction of the photon has some uncertainty. This uncertainty is proportional to the square of the wavelength.

This implies that the concept of a specific direction of a photon breaks down in certain circumstances. When this is not the case, a useful approximation is made by defining the concept of rays, which are physical entities associated with photons. A bundle of rays can be thought of as a set of quasi-plane waves that can have any direction at each point and that propagate in space by following the laws of geometric
optics. The bundle of rays should be considered as a superposition of incoherent waves, in the sense that no interference effect are present and their intensity is the sum of the intensity of each wave without amplitude term.

Specific Intensity (Brightness)

The flux is a measure of the energy carried by **all rays** passing through a given area.

What if we want to follow each ray instead?
Note: in transfer theory (which is the one we're using here) a single ray carries no energy, so we need to consider an infinitesimal amount of energy carried by a set of rays which differ infinitesimally from the given ray.



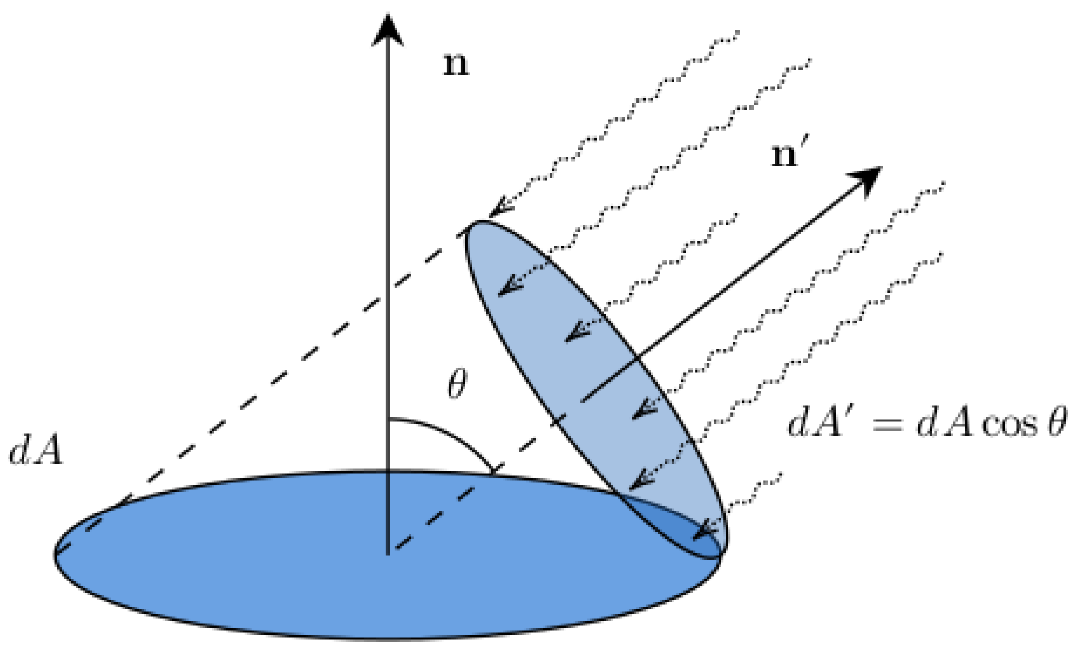
Specific Intensity: energy passing through an area dA normal to the direction of the given ray and consider all rays passing through dA whose direction is within an infinitesimal solid angle of the given ray

$$dE = I_\nu \, dA \, d\Omega \, dt \, d\nu$$

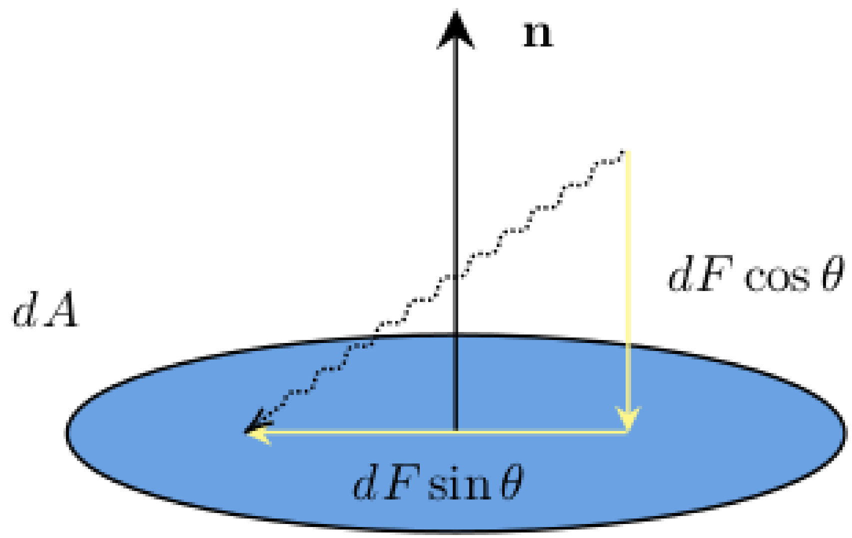
Momentum Flux

Suppose now that we have a radiation field coming from a direction **n'** and construct a small element of area dA at some arbitrary orientation **n** (when we say *a direction **n** and **n'*** we mean "rays within a certain infinitesimal solid angle dOmega", i.e., a bunch of rays). "Flux" is an energy per unit time per unit area:

$$dF_\nu = I_\nu \cos\theta \, d\Omega \rightarrow \frac{dF_\nu}{c} = momentum\ flux$$



cos θ appears in the expression because of the projected area effect



The flux of momentum normal to dA has units of momentum per unit time per unit area which is equivalent to **pressure**.
The component of the momentum flux normal to dA is:

$$\frac{dF_\nu \cos\theta}{c}$$

Note that there is another cos(theta) here because we are selecting only the component normal to dA.

Therefore the pressure exerted by radiation on the area dA is exactly what we have written above:

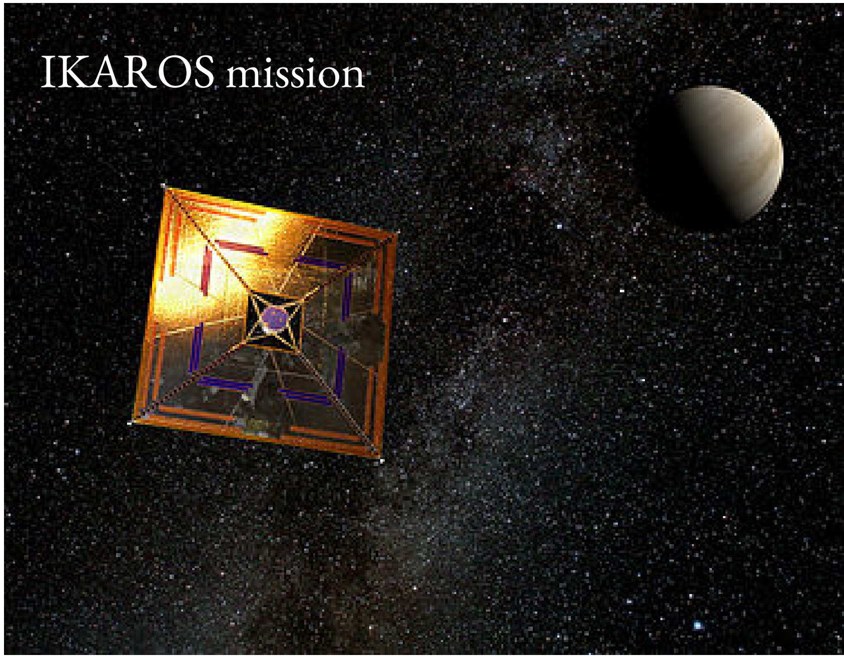
$$dp_\nu = \frac{dF_\nu \cos\theta}{c} = \frac{I_\nu}{c}\cos^2\theta\, d\Omega$$

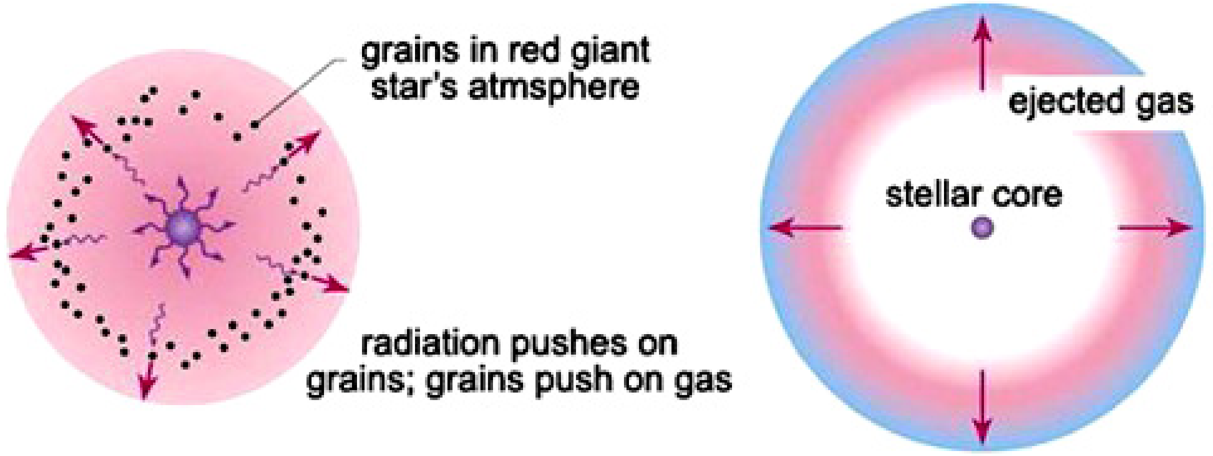
Radiation Pressure

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega \rightarrow [dyne\, cm^{-2}\, Hz^{-1}]$$

RADIATION PRESSURE (dyne/cm2)

$$p = \int p_\nu \, d\nu$$

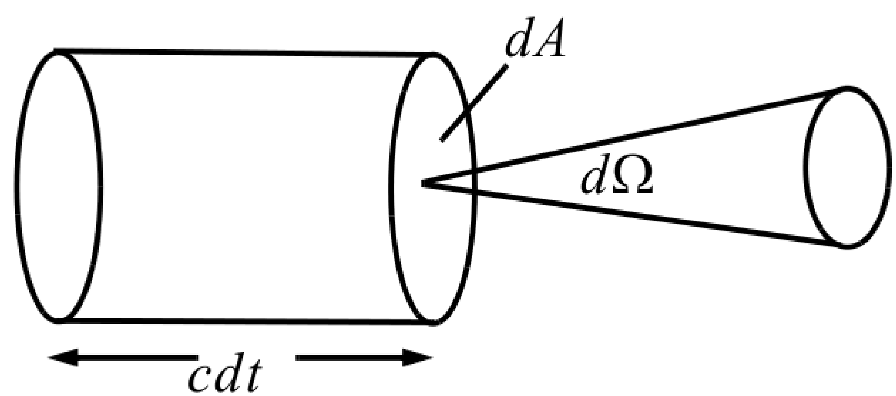




Radiative Energy Density

The *specific energy density* u_ν is defined as the energy per unit volume per unit frequency range. To determine this it is convenient to consider first the energy density per unit solid angle $u_\nu(\Omega)$ via:

$$dE = u_\nu(\Omega)\, dV\, d\Omega\, d\nu = u_\nu(\Omega)\, dA\, c\, dt\, d\Omega\, d\nu$$



Remember what we said before:

$$dE = I_\nu\, dA\, d\Omega\, dt\, d\nu$$

Equating the two:

$$u_\nu(\Omega) = \frac{I_\nu}{c}$$

Radiative Energy Density

Integrate now over the solid angle:

$$u_\nu(\Omega) = \frac{I_\nu}{c} \rightarrow u_\nu = \int u_\nu(\Omega)\, d\Omega = \frac{1}{c}\int I_\nu\, d\Omega = \frac{4\pi}{c} J_\nu$$

where we have defined the *mean intensity* as:

$$J_\nu = \frac{1}{4\pi}\int I_\nu\, d\Omega$$

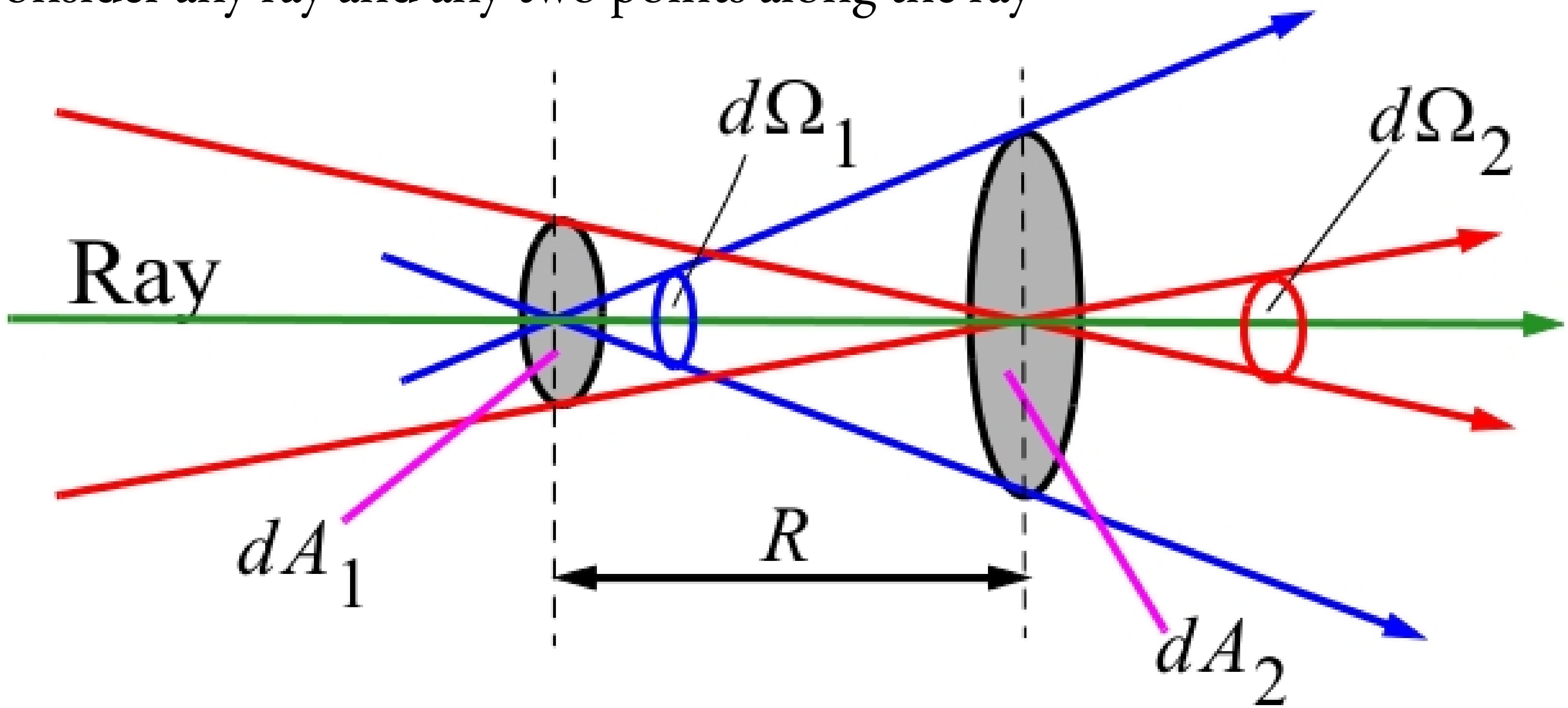
Radiative Energy Density

The total *energy density* **u** is therefore:

$$u = \int u_\nu \, d\nu = \frac{4\pi}{c} \int J_\nu \, d\nu$$

Constancy of Brightness in Free Space

Consider any ray and any two points along the ray



Consider the energy flux through an elementary surface dA1 within solid angle $d\,\Omega_2$ Consider all of the photons which pass through dA1 in this direction which then pass through dA2 . We take the solid angle of these photons to be $d\,\Omega_1$

Constancy of Brightness in Free Space

We use now energy conservation:

$$dE = I_{\nu 1}\, dA_1\, d\Omega_1\, dt\, d\nu_1 = I_{\nu 2}\, dA_2\, d\Omega_2\, dt\, d\nu_2$$

Remember the definition of solid angle:

$$d\Omega_1 = dA_2 / R^2 \qquad\qquad d\Omega_2 = dA_1 / R^2$$

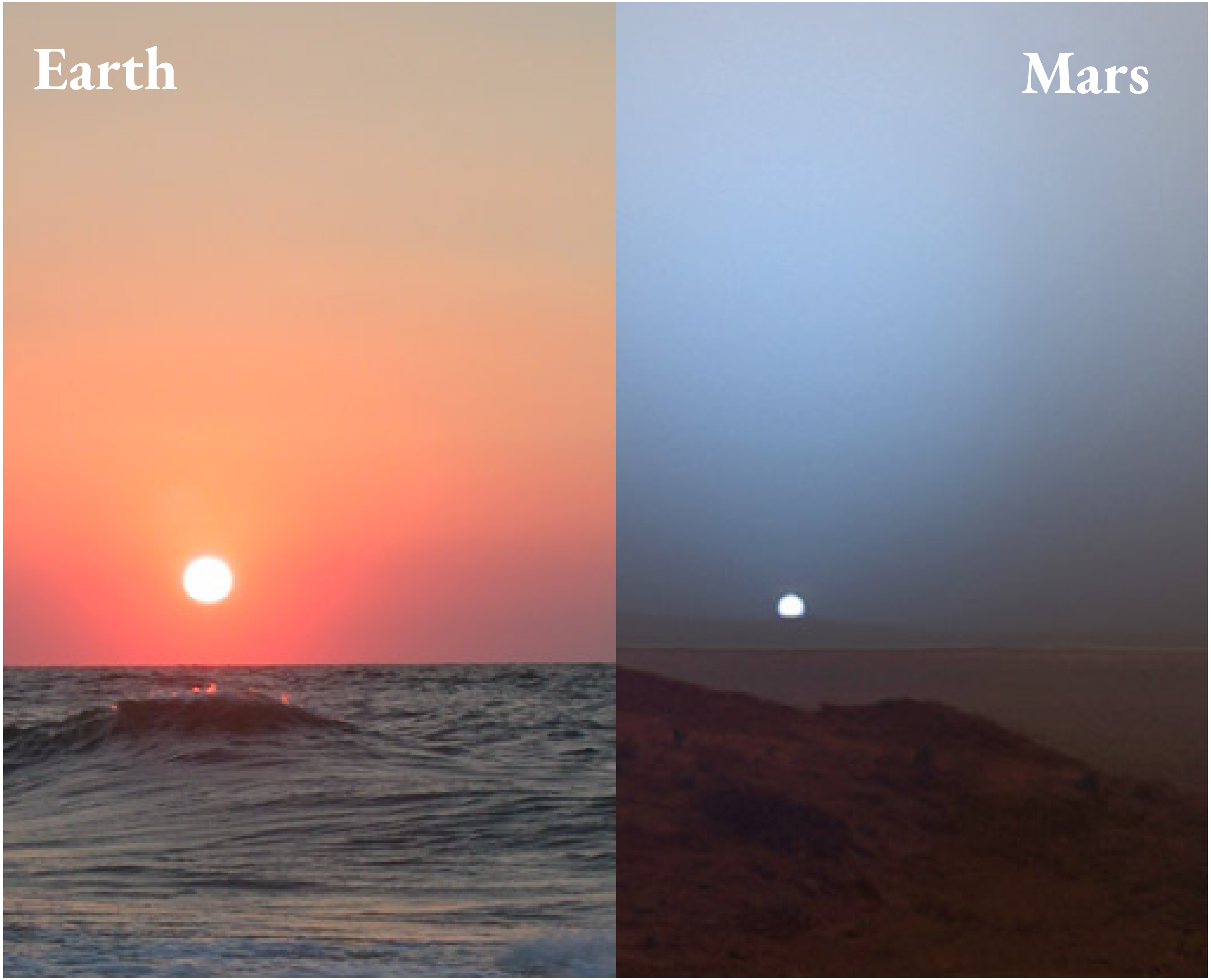
Since the frequency of the ray is the same ($d\nu_2 = d\nu_1$) we have:

$$I_{\nu 1} = I_{\nu 2} = constant$$

Brightness does NOT depend on distance

Do we measure Flux or Brightness?

Do we measure Flux or Brightness?



Do we measure Flux or Brightness?

Answer: it depends on the object!!
If the telescope cannot resolve the object then we measure flux.
If the telescope can resolve the object, then we measure intensity.

Why? Consider the case when the source is unresolved. Now imagine pushing the source to farther distance. As the distance increases, the number of photons falls as r^2 . The flux is measured.

If instead the the source is resolved, then as the source is pushed farther away, more area of the source would be included with the solid angle, which compensates for the the increased distance and the collected number of photons remain the same.

Look at the example of the momentum flux discussed earlier: the presence of the unit solid angle tells us that the specific brightness re-
quires that our bundle of rays are coming from a direction normal to dA within a solid angle $d\Omega$ from the direction $\vec{n}$. Therefore the source emitting the bundle of rays needs to be resolved, which means that different rays crossing the surface area dA and forming our bundle, come from slightly different directions from slightly different points belonging to the source. In other words, a point source cannot have a specific intensity, since all rays come from a single direction. In that case we would talk about specific flux rather than intensity.

What is radiative transfer?

- Radiative transfer is the physical phenomenon of energy transfer in the form of electromagnetic radiation
- The propagation of radiation through a medium is affected by absorption, emission and scattering processes.
- The equation of radiative transfer describes these interactions mathematically.
- Applications apart from astrophysics include optics, atmospheric science, and remote sensing.
- Analytical solutions exist in a few (simple) cases, but more realistic cases need a numerical treatment.

Why do we study transfer theory?

The light we detect arrives at us in two steps:
- first, it is created by some radiative process (e.g., blackbody, synchrotron, etc etc...)
- then it propagates through space where it might be (partially) scattered and absorbed

Scattering, absorption and emission are thus three fundamental steps to generate the light we see.

Transfer theory tells us how the specific intensity of an object is affected by absorption, scattering and emission.

Constancy of Brightness in Free Space

We use now energy conservation:

$$dE = I_{\nu 1}\, dA_1\, d\Omega_1\, dt\, d\nu_1 = I_{\nu 2}\, dA_2\, d\Omega_2\, dt\, d\nu_2$$

Remember the definition of solid angle:

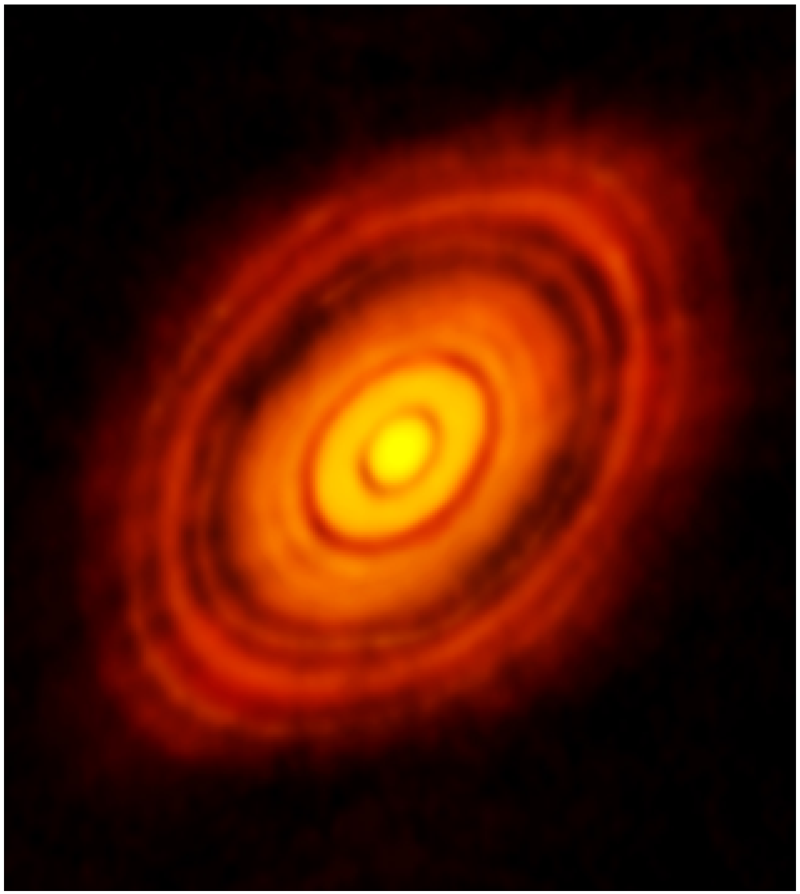
$$d\Omega_1 = dA_2 / R^2 \qquad d\Omega_2 = dA_1 / R^2$$

Since the frequency of the ray is the same ($d\nu_2 = d\nu_1$) we have:

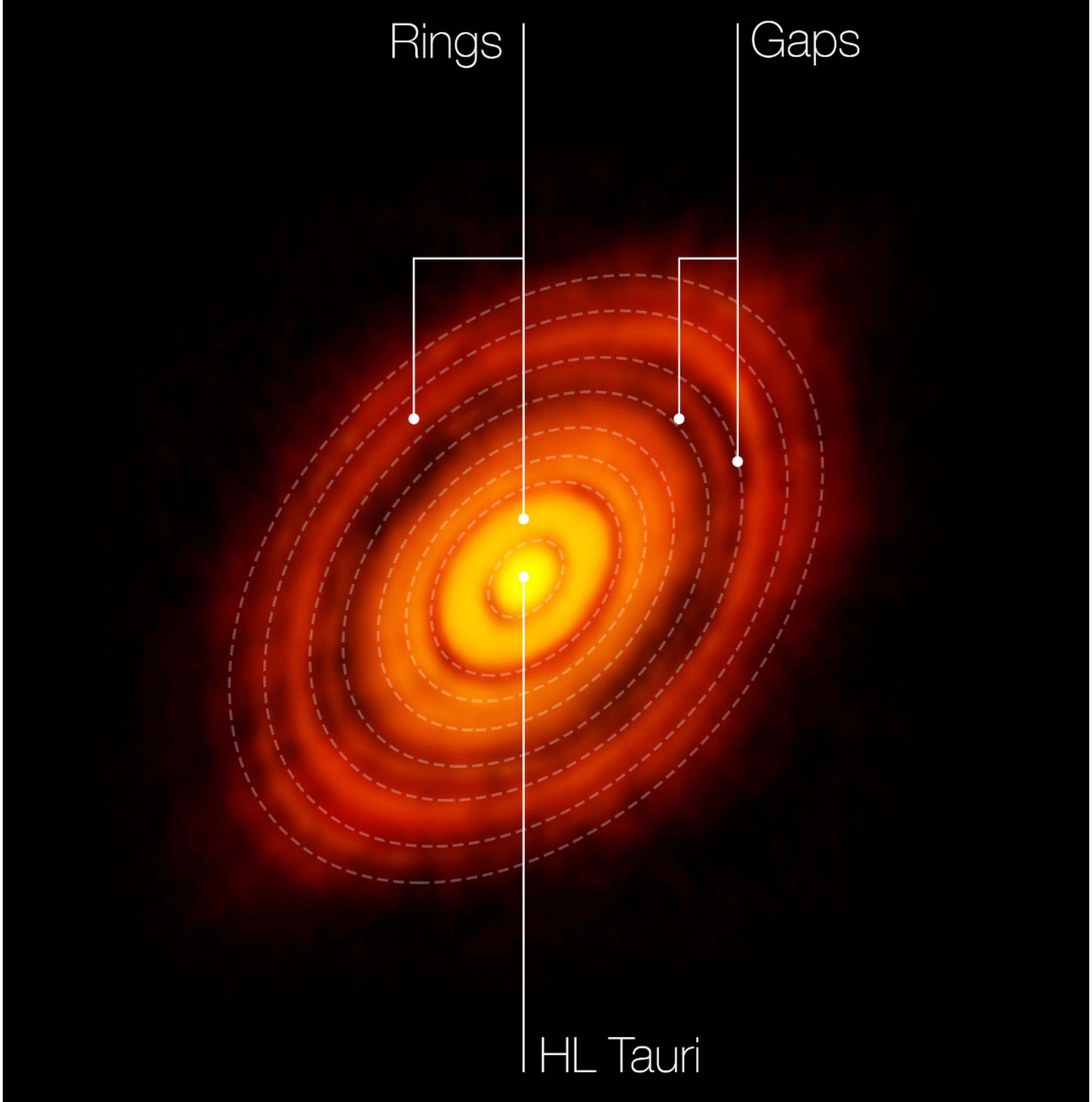
$$I_{\nu 1} = I_{\nu 2} = constant$$

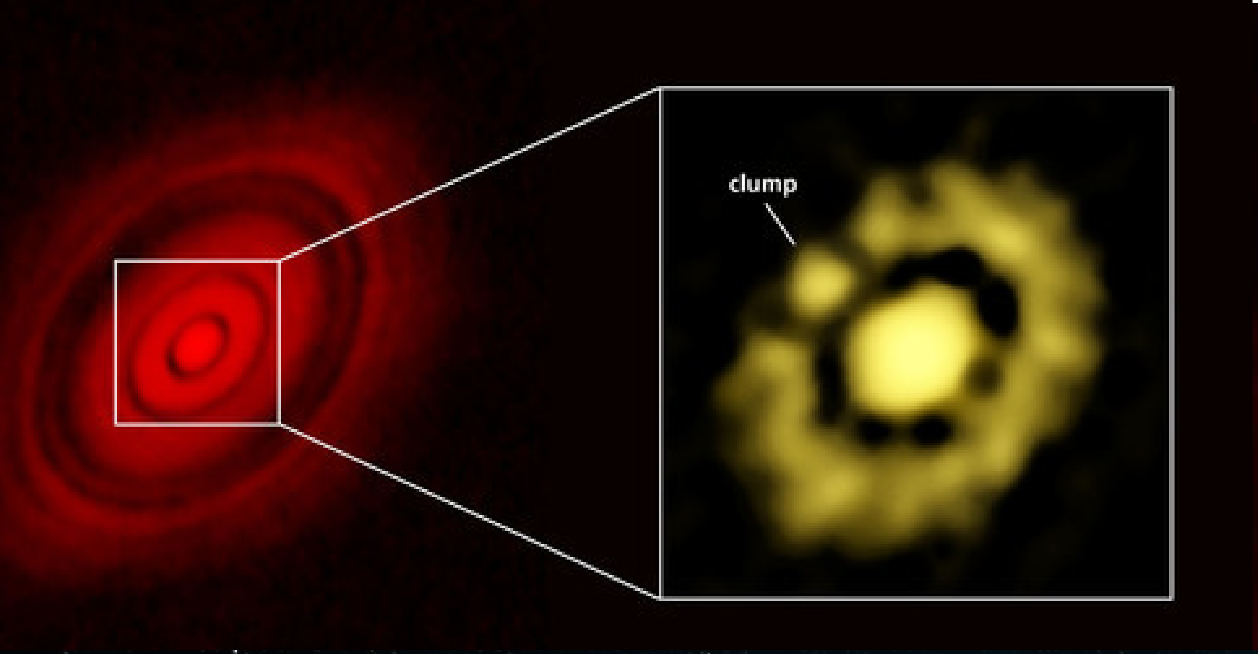
Brightness does NOT depend on distance

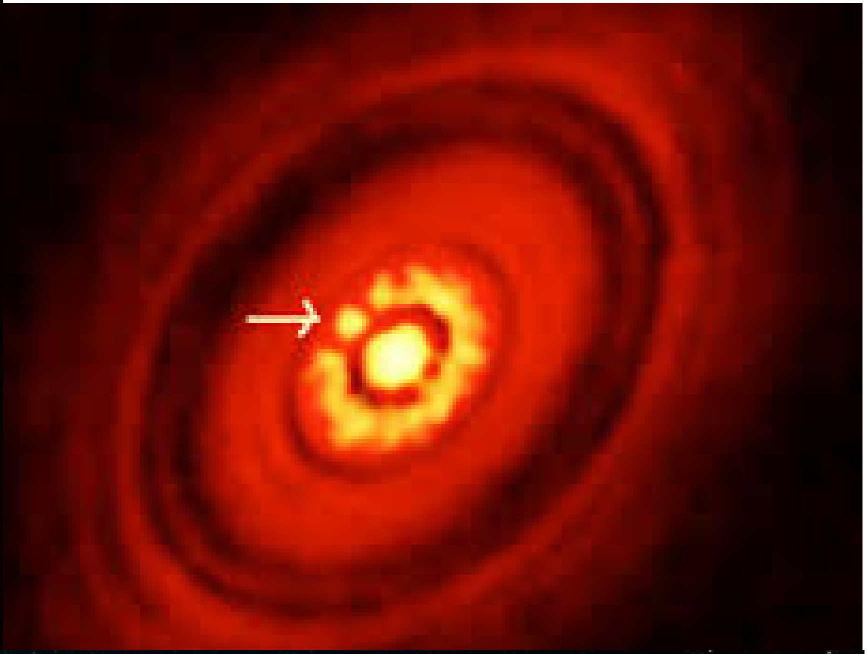
HL Tau

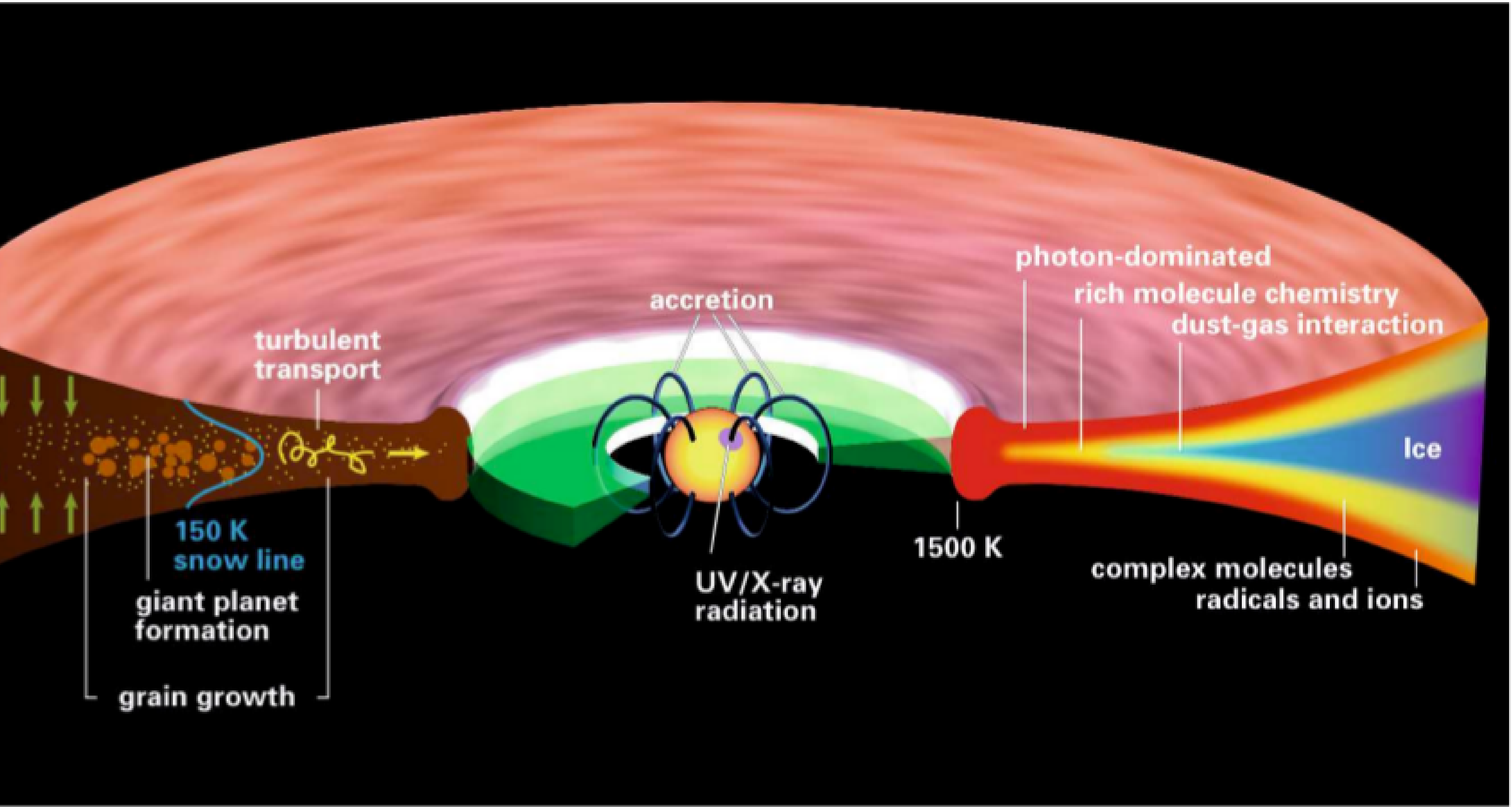


- ALMA radio image of protoplanetary disk around young star
- Rings show up because dust is cleared out by protoplanets
- Resolution 35 microarcsec (penny at 110 km distance)


Rings
Gaps
HL Tauri

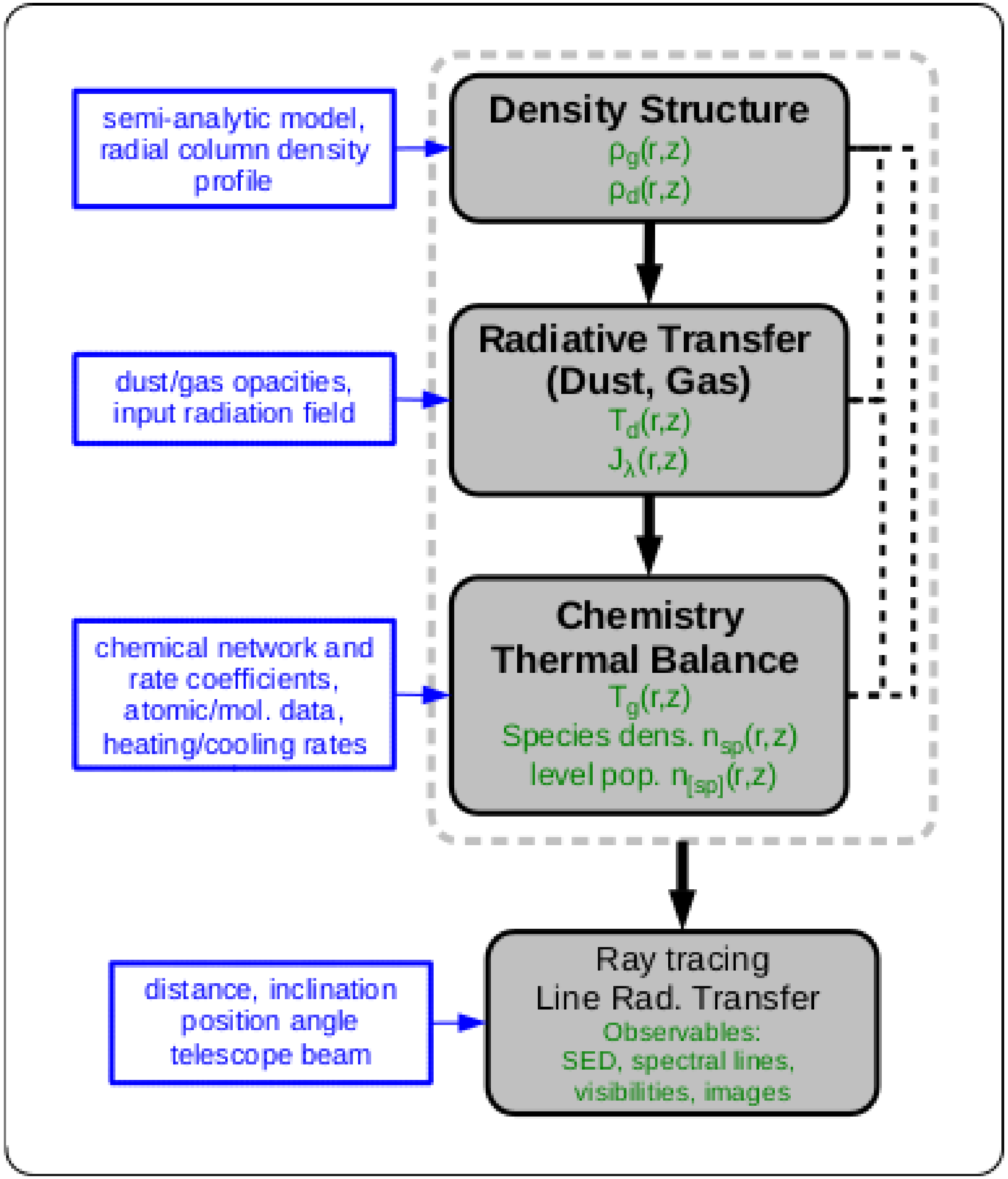

clump





Credit: Henning and Semenov (2013)

ProDiMo - Code overview



Vertical Structure

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{z\mathrm{GM}_*}{(r^2+z^2)^{3/2}}$$

Radiative Transfer

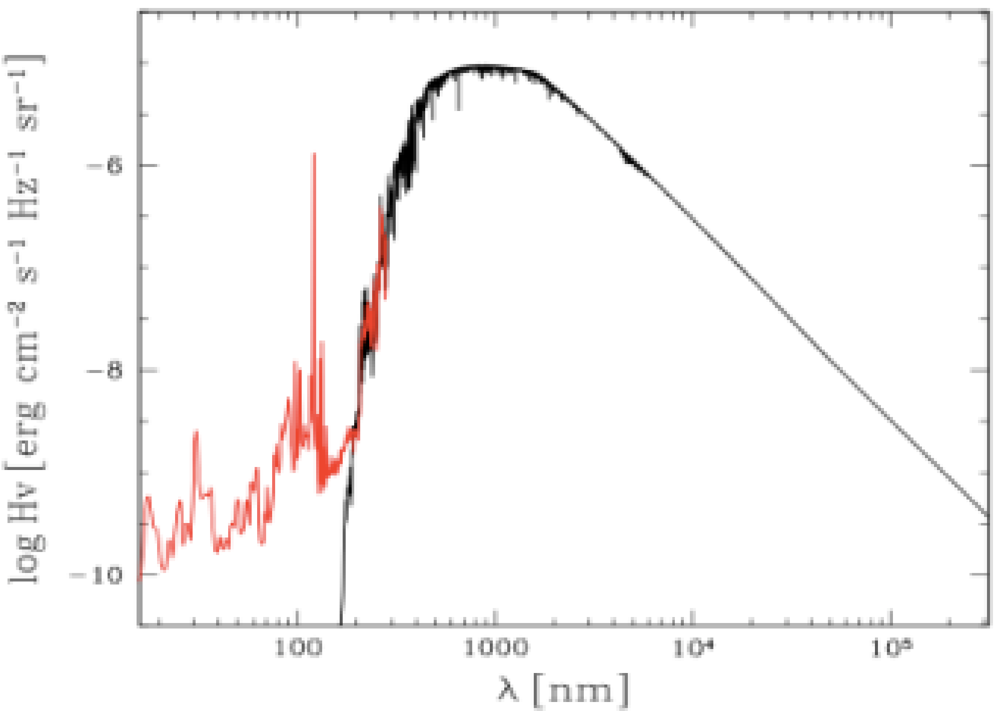
$$\frac{\mathrm{d}I_\lambda}{\mathrm{d}\tau_\lambda} = I_\lambda - S_\lambda$$

Chemistry & Thermal Balance

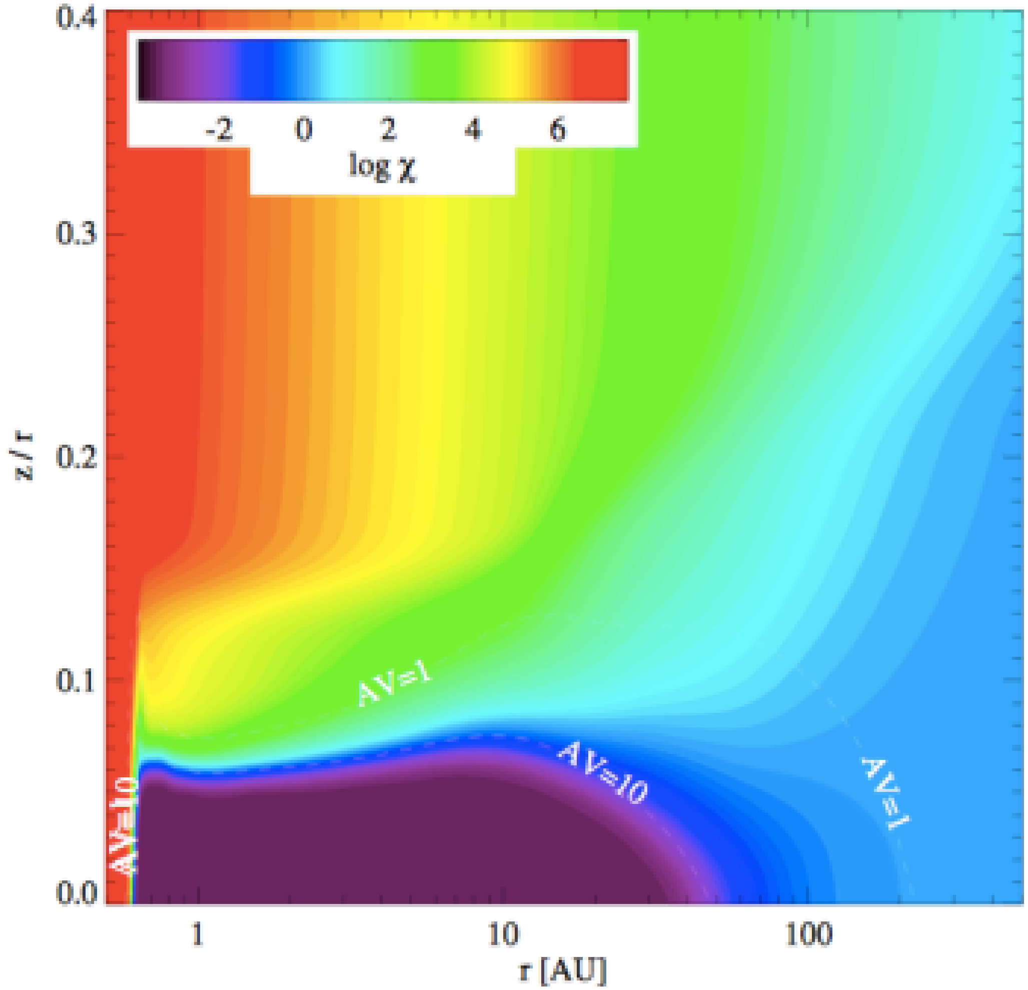
$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = \sum_{jkl} R_{jk\to kl}(T_\mathrm{g})n_j n_k \cdots$$
$$-n_i \sum_{jkl} R_{il\to jk} n_l \ldots$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \sum_k \Gamma(T_\mathrm{g}, n_\mathrm{sp}) - \sum_k \Lambda(T_\mathrm{g}, n_\mathrm{sp})$$

Radiative transfer in the disk



Input spectrum of typical T Tauri star



Radiation field throughout the disk

Why do we need radiative transfer?

- Crucial to determine radiation contribution throughout the object of interest
- It is key in determining the heating and cooling processes, and as a result the density, thermal and chemical structure
- Finally, dust and line radiative transfer will provide dust and emission characteristics to be observed with telescopes.

Radiative transfer

- If a ray passes through a medium, energy can be added or subtracted by emission and absorption
- Therefore: Specific intensity will usually not remain constant when passing through the interstellar medium.

Emission (1)

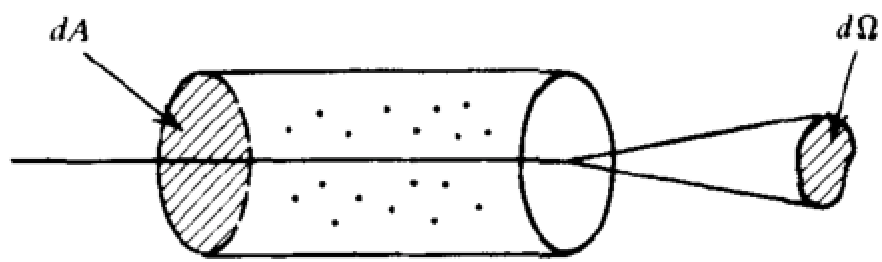
- The spontaneous emission coefficient j is defined as the energy emitted per unit time per unit solid angle per unit volume:
  $dE = j\, dV\, d\Omega\, dt$
- Or when the emission is monochromatic:
  $dE = j_\nu\, dV\, d\Omega\, dt\, d\nu$
- If the emission is isotropic, we can write:
  $j_\nu = 1/(4\pi)\, P_\nu$, with the power per unit volume per unit frequency.

Emission (2)

- In going a distance *d*s, a beam of cross section dA travels through a volume *d*V = *d*A *ds*
- The intensity added to the beam is then: $dI_\nu = j_\nu \, ds$
- Or compare the specific intensity and the emission coefficient:
  j_ν [erg cm^{-3} s^{-1} ster^{-1} Hz^{-1}] to
  I_ν [erg cm^{-2} s^{-1} ster^{-1} Hz^{-1}]

Absorption

- The absorption coefficient α [cm^{-1}] is defined by the following equation: $dI_\nu = -\alpha_\nu I_\nu ds$, representing the loss of intensity in a beam as it travels a distance *ds.*
- This α can be defined by: $\alpha_\nu = n\sigma_\nu$ or $\alpha_\nu = \rho\kappa_\nu$



(a)

Radiation transport

- The decrement of I_ν when passing through a path of length *ds*:
  $dI_\nu = -\alpha_\nu I_\nu ds$
- Inside a source, a contribution to I_ν can be made from emitters. The increment is:
  $dI_\nu = j_\nu ds$
- The basic equation of transport is:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

Simple solutions (1)

- Emission only:

$$\frac{dI_\nu}{ds} = j_\nu \rightarrow I_\nu = I_{\nu,0} + \int_0^S j_\nu ds$$

with S the total emission path.

- The increase in the specific intensity is equal to the emission coefficient integrated along the line of sight

Simple solutions (2)

- Absorption only:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \rightarrow \frac{dI_\nu}{I_\nu} = -\alpha_\nu ds$$

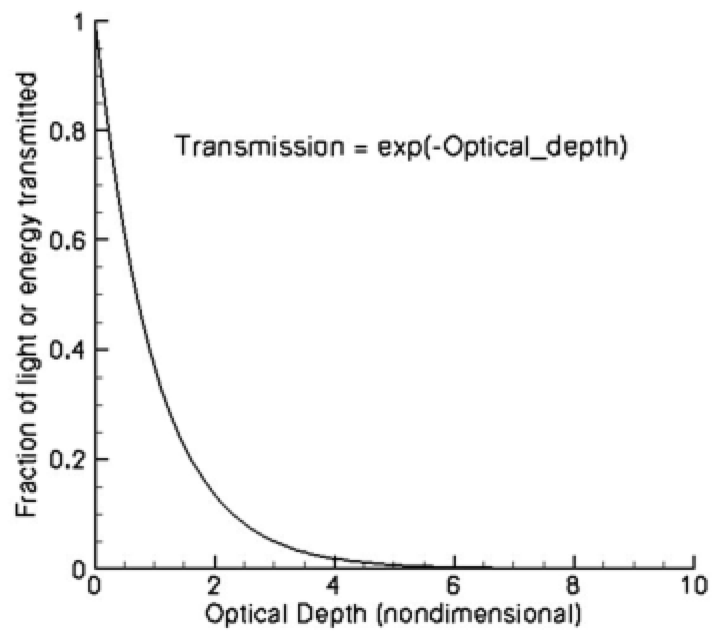
$$I_\nu(s) = I_\nu(s_0)\, e^{-\int_{s_0}^{s} \alpha_\nu(s')ds'}$$

- The brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight

Optical depth (1)

- We now introduce the quantity **optical depth:**

$$d\tau_\nu = \alpha_\nu ds = n\sigma_\nu ds$$



Optical depth (2)

- The intensity decreases as follows:
  $I_\nu = I_{\nu,0} \exp(-\tau_\nu)$

- What follows from this:
  - $\tau=1 \rightarrow 1/e$ (37%)
  - $\tau>>1 \rightarrow$ optically thick
  - $\tau<<1 \rightarrow$ optically thin

Optical depth is connected with more familiar concepts like reflectivity, for example a very good mirror usually reflects 90-95% of light. This means that some fraction (5-10%) of radiation is “absorbed” (it is actually transmitted).
You can see this effect very clearly when you have multiple reflections in a mirror.
The brightness of the reflected object goes down very quickly.

You can check this at home, but be careful: the effect you see with your eye is not the same that you record on camera. Why?

The "surface" of the Sun

The "surface" of the Sun is usually defined as the location in the photosphere where the optical depth is equal to 2/3.
This is just a definition, and it is used because about 50% of photons we see are coming from this region.

Transport equation and source function

- The rewritten equation of transport becomes

$$\frac{dI_\nu}{\alpha_\nu ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu} \rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{\alpha_\nu}$$

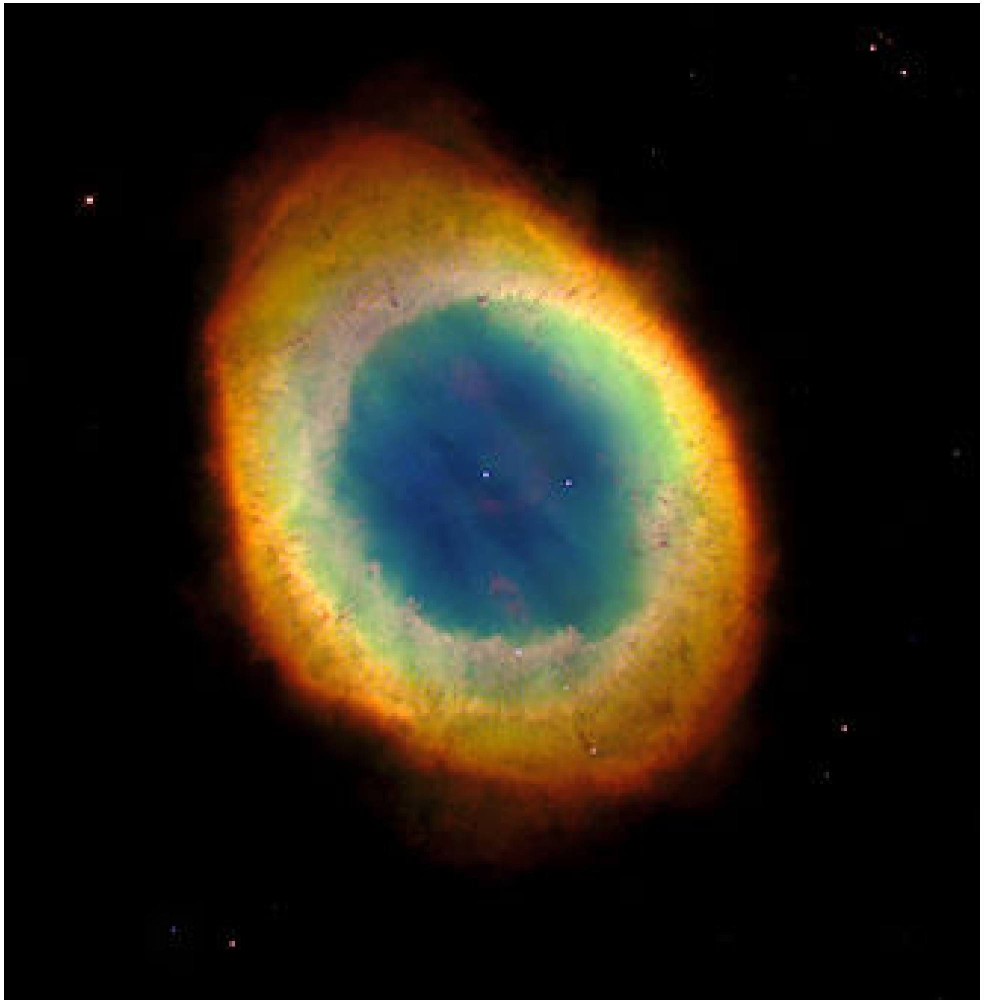
- We define S_ν the source function as follows

$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

(The source function can be thought in a way as the local input of radiation.
For example when optical depth >>1 then source function ~ specific brightness
which means that little is contributed to the to the specific brightness from matter far from the location of interest).

Light comes from a white dwarf illuminating the surrounding material

Helix Planetary Nebula
(remnant of a low mass star)
Distance: 220 pc
Size: about 0.8 pc across



Ring Planetary Nebula
(remnant of a low mass star)
Distance: 700 pc
Size: about 0.4 pc across

Equation of transfer

- This yields the format solution of the EOT:

$$I_\nu(\tau_\nu) = I_{\nu,0}\, e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

- When S_ν constant:
  $I_\nu(\tau_\nu) = I_{\nu,0}\exp(-\tau_\nu) + S_\nu(1-\exp(-\tau_\nu))$
- $\tau >> 1$: $I_\nu \rightarrow S_\nu$
- $\tau << 1$: $I_\nu \rightarrow I_{\nu,0} + S_\nu \tau_\nu$

A special case

- When S_ν is constant throughout the source, this can be rewritten as:

$$I_\nu(\tau_\nu) = I_{\nu,0}\, e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right)$$

Question: What is the intensity of this source for small and large optical depth when it has size R?

Answer

- If $I_{\nu,0} = 0$ then $I_\nu(\tau_\nu) = \frac{j_\nu}{\alpha_\nu}\left(1 - e^{-\tau_\nu}\right)$
- A little trick. First, we multiply by the source size s=R:

$$I_\nu(\tau_\nu) = \frac{j_\nu R}{\alpha_\nu R}\left(1 - e^{-\tau_\nu}\right) = j_\nu R \left(\frac{1 - e^{-\tau_\nu}}{\tau_\nu}\right)$$

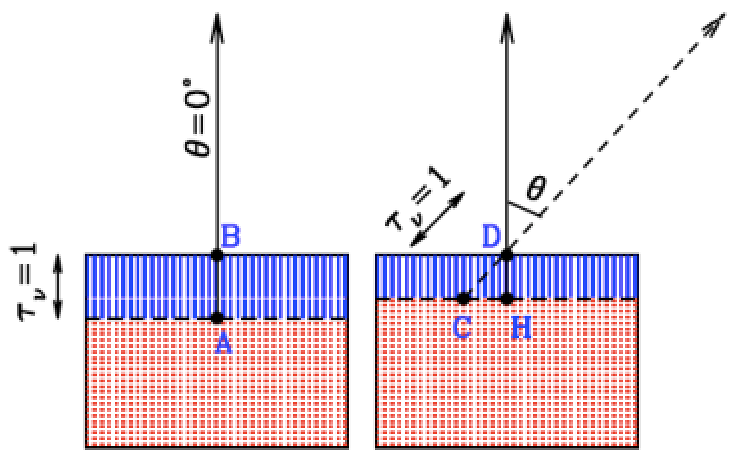
- Optically thin (τ << 1): 1 – exp(-τ) = 1 – 1 + τ = τ

→ $I_\nu(\tau_\nu) = j_\nu R$

- Optically thick (τ >> 1): $I_\nu(\tau_\nu) = \frac{j_\nu R}{\tau_\nu}$

The cos(θ) law

- We often hear the expression that radiation from an optically thick source comes from its surface
- We do mean that the emission we see is emitted from a layer with $\tau = 1$.
- The emitting volume the observer sees depends on inclination.



Mean free path (1)

- The mean free path is the average distance traveled by a photon before being absorbed.
- The probability of a photon to travel at least an optical depth τ_ν is $\exp(-\tau_\nu)$. The mean optical depth is thus unity:

$$\langle \tau_\nu \rangle \equiv \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$$

- In a homogeneous medium, the average distance traveled is defined as l_ν:

$$\langle \tau_\nu \rangle = \alpha_\nu l_\nu = 1 \quad \text{or} \quad l_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}$$

Mean free path (2)

- A source with radius R and total optical depth $\tau > 1$ has a mean free path:

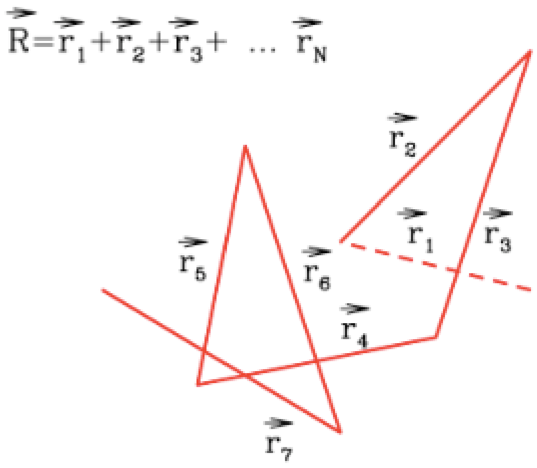
$$\ell_\nu = \frac{1}{\sigma_\nu n} = \frac{R}{\sigma_\nu n R} = \frac{R}{\tau_\nu}$$

Scattering effects: random walks

- Assume a photon that interacts through scattering inside a source R and optical depth $\tau > 1$
- How many times does it scatter before escaping?
- How much time does it take?

Random walks

- The total net displacement after N scatterings is: $R = r_1 + r_2 + r_3 + \ldots + r_N$
- If we want the distance |R| traveled by a typical photon we need to calculate the square displacement:



$$
\begin{aligned}
\langle \vec{R}^2 \rangle &= \langle \vec{r}_1^{\,2} \rangle + \langle \vec{r}_2^{\,2} \rangle + \langle \vec{r}_3^{\,2} \rangle \\
&+ 2\langle \vec{r}_1 \cdot \vec{r}_2 \rangle + 2\langle \vec{r}_1 \cdot \vec{r}_3 \rangle + \ldots.
\end{aligned}
$$

Random walks

- The cross products vanish for isotropic scattering

$$\langle \vec{R}^2 \rangle = N \langle \vec{r}_i^2 \rangle = N\ell^2 \to \sqrt{\langle \vec{R}^2 \rangle} = \sqrt{N}\ell$$

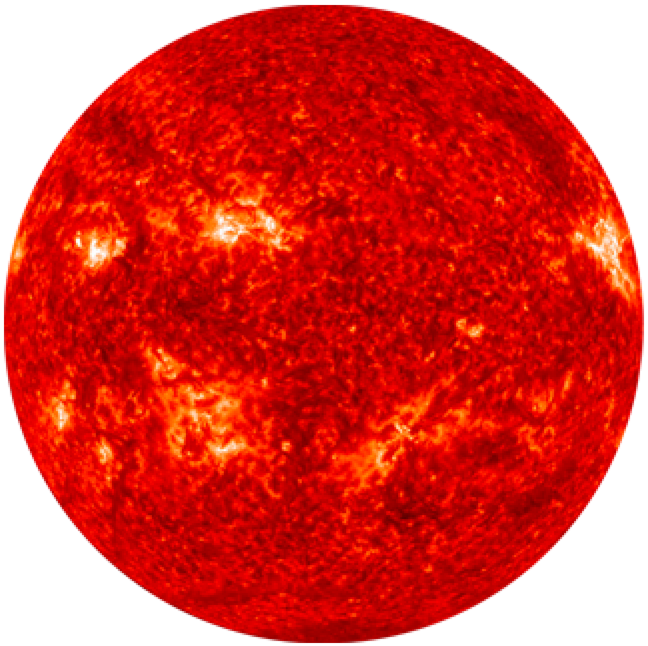
- Question 1:

$$\sqrt{N} = \frac{R}{\ell} = R\sigma n = \tau \to N = \tau^2$$

- Question 2:

$$t_{\rm tot} = N t_1 = \tau^2 \frac{\ell}{c} = \tau^2 \frac{R}{R\sigma n c} = \frac{R}{c}\frac{\tau^2}{\tau} = \tau \frac{R}{c}$$

How long before light escapes the Sun?



Sun's mean density: 1.4 g/cm3

Assume Thomson scattering: $\sigma_{\mathrm{T}} = 6.7 \times 10^{-25}\ \mathrm{cm}^2$

$$R\odot = 7 \times 10^{10}\ \mathrm{cm}$$

$$m_{\mathrm{H}} = 1.66 \times 10^{-24}\ \mathrm{g}$$

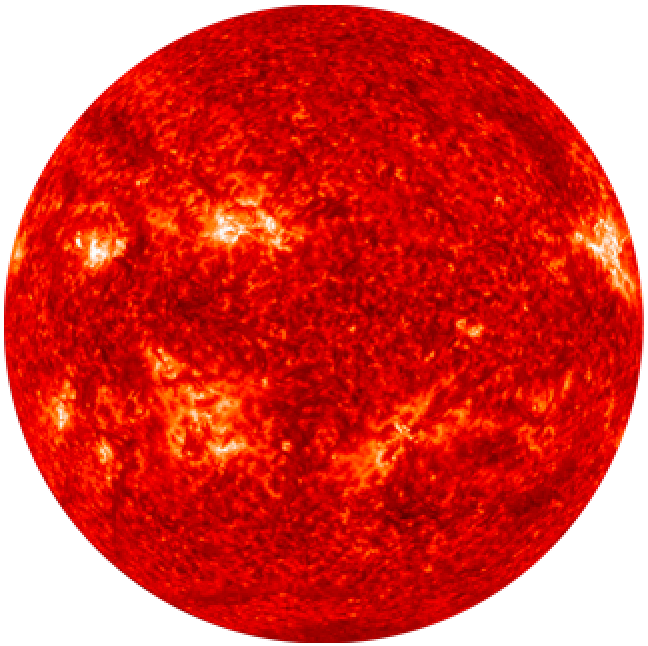
Assume Sun is made by pure hydrogen

Then: $n = \rho / m_{\mathrm{H}} \quad 8.4 \times 10^{23}\ \mathrm{cm}^{-3}$

Mean free path: $\ell = \dfrac{1}{n\sigma_{\mathrm{T}}}$ ~ 1 cm. The time required to escape the Sun will be the time to travel one mean free path times N scatter events. Since each mean free path is done at speed *c*, we have:

$$t = N\frac{\ell}{c}.$$

How long before light escapes the Sun?



The Sun is clearly optically thick, so $N \approx \tau^2$

Therefore: $\tau = R_\odot / \ell$ and by squaring it we get:

$$t = \left(\frac{R_\odot}{\ell}\right)^2 \frac{\ell}{c}$$

By plugging the numbers in, we obtain a number of the order of a few thousand years.

Why The Sky is Blue and the Sun is Red at Sunset

We know that the Rayleigh scattering has a strong dependence on frequency.

Indeed the cross-section of the scattered light scales in the following way:

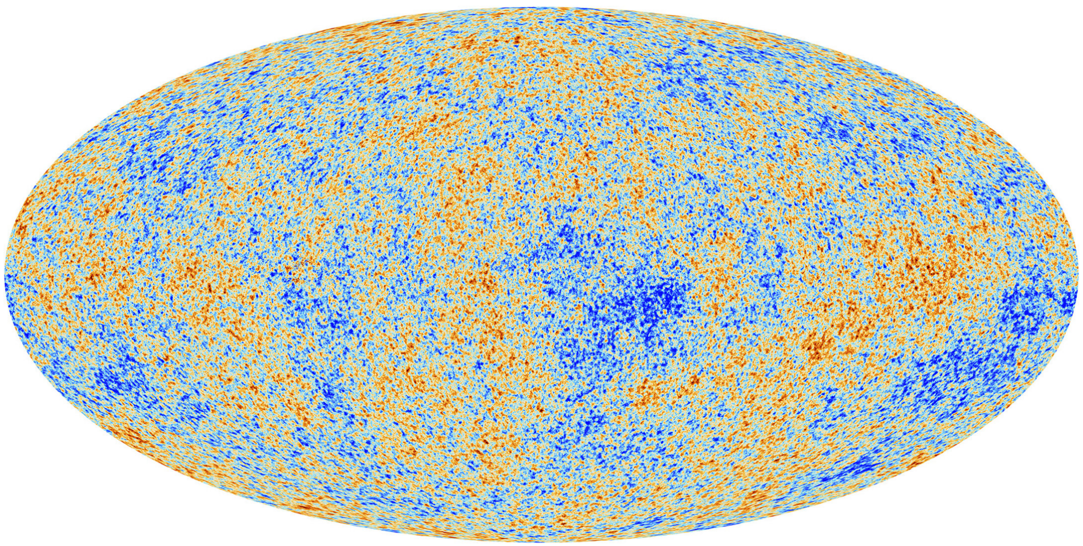
$$\sigma_\nu \propto \nu^4$$

The scattering therefore is very large for blue light which has a much larger random walk in the sky. Indeed the mean free path for blue light:

$$l_\nu = \frac{1}{\sigma_\nu n}$$

is very small and the number of scatterings N is very large and so is the optical depth. Red light instead has a much larger mean free path, N is small and the region of the sky affected by scattering is smaller.

Example: relation between flux and brightness



The Cosmic Microwave Background is an isotropic radiation field that permeates the whole universe.

The specific intensity (or specific brightness) at 100 GHz is 10^-15 erg/cm2/s/sr/Hz

What is the specific flux observed by the Planck Satellite at that frequency?

Hint:
Flux has units of erg/cm2/s
Specific Flux has units of erg/cm2/s/Hz